Rudin Ch3 - L^P spaces

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1 Notes

1.1 Convex functions and inequalities

3.1 Def A function φ on (a, b), where $-\infty \leq a \leq b \leq \infty$ is convex if

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

For all $a < x, y < b$ and $0 \leq \lambda \leq 1$.

3.2 Thm is convex implies φ is continuous.

3.3 Thm (Jensen's Inequality Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\varphi \circ f) d\mu$$

3.4 Def If p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then p and q are *conjugate exponents.*

3.5 Thm Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable functions on X, with range $[0, \infty]$, then

$$\int_X fg d\mu \leq \left\{\int_X f^p d\mu\right\}^{1/p} \left\{\int_X g^q d\mu\right\}^{1/q}$$

and

$$\left\{\int_X (f+q)^p d\mu\right\}^{1/p} \leq \left\{\int_X f^p d\mu\right\}^{1/p} + \left\{\int_X g^p d\mu\right\}^{1/p}$$

The first one is Holder's, and the second is Minkowski. If $p = q = 2$, then Holder's is known as the Schwarz inequality.

1.2 L^P spaces

3.6 Def If $0 < p < \infty$ and if f is a complex measurable function on X, define

$$||f||_p = \left\{ \int_X |f|^p d\mu \right\}^{1/p}$$

Let $L^p(\mu)$ consist of all f for which

$$||f||_p < \infty$$

and we call $||f||_p$ the L^p norm of f. On $\mathbb{R}^k$, it is $L^p(\mathbb{R}^k)$. If μ is a counting measure on A, the L^p space is called $\ell^p(A)$. An element is a complex sequence $x = \{\zeta_n\}$, and

$$||x||_p = \left\{ \sum_{n=1}^{\infty} |\zeta_n|^p \right\}^{1/p}$$

3.7 Def Suppose $g : X \to [0, \infty]$ is measurable. Let S be the set of all real α such that

$$\mu(g^{-1}((\alpha, \infty])) = 0$$

If $S = \varnothing$, put $\beta = \infty$. If $S \neq \varnothing$, put $\beta = \inf S$. Since

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right)$$

and since the union of a countable collection of sets of measure 0 has measure 0, we see that $\beta \in S$. We call β the essential supremum of g. The members of $L^\infty(\mu)$ are called the essentially bounded measurable functions on X.

It follows that $|f(x)| \le \lambda$ holds for almost all x iff $\lambda \ge ||f||_\infty$.

3.8 Thm If p and q are conjugate exponents, $1 \le p \le \infty$, and if $f \in L^P(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and

$$||fg||_1 \le ||f||_p ||g||_q$$

3.9 Thm Suppose $1 \le p \le \infty$ and $f \in L^p(\mu)$, $g \in L^p(\mu)$. Then $f + g \in L^p(\mu)$, and

$$||f + g||_p \le ||f||_p + ||g||_p$$

3.10 Remarks Fix p, $1 \le p \le \infty$. If $f \in L^p(\mu)$ and α is a complex number, it is clear that $\alpha f \in L^p(\mu)$. In fact,

$$||\alpha f||_p = |\alpha| ||f||_p$$

which shows that $L^P(\mu)$ is a vector space. We note that this is a complete metric space and that, if a sequence $\{f_n\}$ converges to f in $L^p(\mu)$ if every $\epsilon > 0$ then there corresponds an integer N s.t. $||f_n - f_m|| < \epsilon$. We call $\{f_n\}$ a Cauchy sequence in $L^p(\mu)$.

3.11 Thm $L^p(\mu)$ is a complete metric space, for $1 \leq p \leq \infty$ and for every positive measure μ.

3.12 Thm If $1 \leq p \leq \infty$ and if $\{f_n\}$ is a Cauchy Sequence in $L^p(\mu)$, with limit f , then $\{f_n\}$ has a subsequence which converges pointwise ae to $f(x)$.

3.13 Thm Let S be the class of all complex, measurable, simple functions on X s.t.

$$\mu(\{x : s(x) \neq 0\}) < \infty$$

If $1 \leq p < \infty$, then S is dense in $L^p(\mu)$.

1.3 Approximation by Continuous Functions

3.14 Thm For $1 \leq p < \infty, C_c(X)$ is dense in $L^p(\mu)$.

3.15 Remarks $L^p(\mathbb{R}^k)$ is a completion of the metric spaces which is obtained by endowing $C_c(\mathbb{R}^k)$ with the L^p-metric.

3.16 Def A complex function f on a locally compact Hausdorff space X is said to vanish at infinity if to every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x) < \epsilon$ for all x no in K.

The class of all continuous f on X which vanish at infinity is called $C_0(X)$. $C_c(X) \subset C_0(X)$ and they coincide if X is compact, at which point they are called $C(X)$.
3.17 Thm If X is a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$, relative to the metric defined by the supremum norm

$$||f|| = \sup_{x \in X} |f(x)|$$

2 Problems

Problem 1 Let the functions be denoted as $\{\varphi_\alpha\}$ and let φ_a be the function such that

$$\varphi_a((1-\lambda)x + \lambda y) = \sup_{\alpha} \varphi_\alpha((1-\lambda)x + \lambda y)$$

We can see that

$$\varphi_a((1-\lambda)x+\lambda y) \leq (1-\lambda)\varphi_a(x)+\lambda\varphi_a(y) \leq (1-\lambda)\sup_\alpha \varphi_\alpha(x)+\lambda \sup_\alpha \varphi_\alpha(y)$$

So convexity holds for the supremum. For pointwise limits, this also trivially holds as it holds for $n \to \infty$.

We note that this not necessarily holds for infinums. For example, $y = x^4$ is convex and $y = x^2 + 1$ shows that the minimum doesn't hold. This means that the upper limit (which is composed of infinum of supremum functions) is convex (since the functions are decreasing). The lower limit is not necessarily the case, since we can't guarantee that the infinums $b_1, b_2 \ldots$ are convex.

Problem 3 We note that this proves convexity for each median point. Therefore, we can define a countable sequence $x, y, \alpha_1, \alpha_2, \ldots$ where $\alpha_i \in \{x, y\}$ where each α is the point which brings the successive sum closer to a value λ. We can show that

$$\varphi(\frac{kx+(2^n-k)y}{2^n}) \leq \frac{k}{2^n}\varphi(x)+\frac{2^n-k}{2^n}\varphi(y)$$

Since this is convex, then $\alpha_n : n \to \infty$ is convex and we find that

$$\varphi(\lambda x+(1-\lambda)y) \leq \lambda\varphi(x)+(1-\lambda)\varphi(y)$$

Problem 5

(a) We let $\varphi = x^{\frac{s}{r}}$, which is convex and $\frac{s}{r} > 1$, and apply Jensen's inequality. Note that we can do this because $|f|^r$ is a real valued functions, $0 < f < \infty$ and $\mu(X) = 1$. We have

$$\left(\int_X |f|^r d\mu\right)^{s/r} \leq \int_X |f|^s d\mu$$

Which shows that

$$\left(\int_X |f|^r d\mu\right)^{1/r} \leq \left(\int_X |f|^s d\mu\right)^{1/s}$$

If we set $s \to \infty$, then this holds for $0 < r < s \leq \infty$.

(b) This only happens when Jensen's inequality is strict. This in turn only happens when $|f|$ is constant *a.e.*.

(c) $L^r(\mu) \supset L^s(\mu)$ obviously since $||f||_r$ is bounded by $||f||_s$. These two spaces contain the same functions if there is an equality, and this occurs when the equalities are strict, namely when $|f|$ is constant *a.e.*.

(d) We notice that Jensen's inequality is reversed when our function φ is concave. Taking $\varphi = \log$, we find that

$$\log\left(\int_X |f|^p d\mu\right)^{1/p} \geq \frac{1}{p}\log\int_X |f|^p d\mu = \int_X |f| d\mu$$

So, since our value is decreasing as $p \to 0$, it follows that

$$\lim_{p\to 0}||f||_p = \exp\left\{\int_X \log|f| d\mu\right\}$$

Problem 7 We know that, for $0 < r < s$ and $\mu(X) = 1$, $L^r(\mu) \supset L^s(\mu)$. Further calculation (shown below for cases $\mu(X) > 1$) shows that $\mu(X) < 1$ this relation holds

Similarly, if $\mu(X)\infty$, we note that we can rescale μ to $\mu' = \frac{\mu'}{\mu(X)}$. We find that

$$\left(\int_X |f|^r d\mu'\right)^{s/r} \leq \int_X |f|^r d\mu'$$

and we find that, if $||f||_s$ and $||f||_r$ are $< \infty$, we have

$$\frac{1}{a^{s/r}}\left(\int_X |f|^r d\mu\right)^{s/r} \leq \frac{1}{a}\int_X |f|^r d\mu$$

$$\int_X |f|^r d\mu > \int_X |f|^s d\mu$$

and that implies that $L^r(\mu) \subset L^s(\mu)$.

However, for $1 < \mu(X) < \infty$, depending on the relative values of f, s and r, we can't define relation that holds between $\int_X |f|^r d\mu$ and $\int_X |f|^s d\mu$. Here that means there is no containment relation between $||f||_r$ and $||f||_s$.

Problem 9 We note that

$$\int_X |f|^p d\mu \leq \Phi(p)^p$$

for all Φ since Φ tends to infinity, while $\sup|f|$ is stable, and $\int_X |f|^p d\mu \leq \sup_{(0,1)} \int_X |f|^p d\mu$.

Problem 11 We note that

$$||f||_1 \leq ||f||_2$$

And so

$$\int_\Omega f d\mu \cdot \int_\Omega g d\mu \geq ||f||_2 \cdot ||g||_2 \geq \int_\Omega fg d\mu \geq 1$$

Problem 13 Equality in 3.8 generally occurs when $e^{s/p+t/q} = p^{-1}e^s + q^{-1}e^t$. This only occurs when $s/p = t/q$, so this happens specifically when

$$p(\log f - \log ||f||_p) = q(\log g - \log ||g||_q)$$

In the 1 or ∞ case, we note that note that this means that $f = ||f||_p$ and $g = ||g||_q$.

For equality in 3.9, this occurs when Holder's works for the proof values.

Problem 15 Let $f = \sum_{n=1}^{\infty} a_n \chi_{[n,n+1]}$. If we set

$$F(x) = \frac{1}{x}\left(\sum_{n=1}^{\lfloor x \rfloor} a_n + (x - \lfloor x \rfloor) a_{n+1}\right)$$

which means that $F(x) = \frac{1}{x}\int_0^x f(t)dt$. It follows that

$$||F||_p \leq \frac{p}{p-1}||f||_p$$

If we integrate with respect to the Lebesgue measure, then we obtain the desired inequality for the whole values if we assume $a_n \geq a_{n+1}$. This is because $F(\lceil x \rceil) \leq F(x)$

For the more general case, we note that a_n not being decreasing implies that the sum is less than the maximial value (when it is decreasing).

Problem 17
(a) When $p > 1$, then we notice that

$$|\alpha - \beta|^p \leq (|\alpha| + |\beta|)^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p)$$

since Jensen's inequality shows us that x^p is convex implies that

$$\left(\frac{|\alpha| + |\beta|}{2}\right)^p \leq \frac{|\alpha|^p + |\beta|^p}{2}$$

when $p \leq 1$, we notice that

$$|\alpha - \beta|^p = \frac{1}{|\alpha - \beta|^{-p}} \geq \frac{1}{(|\alpha + \beta|)^{-p}} \geq \frac{2^{1-p}}{|\alpha|^p + |\beta|^p} \geq \frac{1}{|\alpha|^p + |\beta|^p}$$

(b)

(i) We note that $\limsup \int_A |f_n|^p d\mu = -\limsup - \int_A |f_n|^p d\mu$ and we can find that

$$\liminf_{n\to\infty} \int_B |f_n|^p d\mu \geq \int_B \liminf_{n\to\infty} |f_n|^p d\mu$$

and this finds that this also holds almost everywhere. We have

$$\liminf_{n\to\infty}\int_A |f_n|^p d\mu \geq \int_A \liminf_{n\to\infty} |f_n|^p d\mu$$

Rearranging this equality, we find that

$$\liminf_{n\to\infty} -\int_A |f_n|^p d\mu + \int_A |f|^p d\mu \geq 0 \implies \limsup_{n\to\infty}\int_A |f_n|^p d\mu \leq \epsilon$$

(ii) We find that

$$\begin{aligned}\int_X h_n d\mu &\leq \liminf_{n\to\infty}\int_X \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p d\mu \\ &= \int_X 2\gamma_p |f|^p d\mu - \limsup_{n\to\infty}\int_X |f_n - f|^p d\mu\end{aligned}$$

and note that, as $n \to \infty$, we have

$$\int_X h_n d\mu = \int_X h_n - \limsup_{n\to\infty}\int_X |f_n - f|^p d\mu$$

and we find that $\int_X |f_n - f|^p d\mu = 0 \implies ||f_n - f||_p = 0$.

(c) We will not have the fact that $\int_X |f_n|^p d\mu = \int_X |f|^p d\mu$, so we can't show part (ii).

Problem 19 The essential range R_f is the set of all complex numbers which are part of the range or limits of the ranges. The complement is

$$\mu(\{x : |f(x) - w| < \epsilon\}) = 0$$

Clearly this is open because every neighborhood of radius $\frac{\epsilon}{2}$ is entirely within this set. Furthermore, R_f itself must be compact because $f \in L^\infty(\mu) \implies |f(x)| \leq \lambda$ a.e. Therefore, R_f is compact.

We note that $||f||_\infty \leq \lambda \iff |f(x)| \leq \lambda$ for a.e. x. Since $|f(x)| \leq \lambda$ a.e., it follows that R_f is contained in the neighborhood at the origin with radius $||f||_\infty$.

Because A_f is the set of all possible averages, it follows that it is contained in R_f. Further examination shows that $A_f = R_f$ because all ranges are contained in A_f trivially, and all limit points of ranges are contained in A_f as we can cause A_f to approach each limit.

A_f is therefore closed (since it compact). To check convexity, we need to show that every points between two points in A_f lies in A_f. We clearly see that, for regular measures this holds. However, for measures that are not regular, we have

$$\frac{1}{\mu(E)}\int_E f d\mu \neq \inf_{E\subset V}\frac{1}{\mu(V)}\int_V f d\mu$$

If f is instead just a $L^1(\mu)$, then we can no longer prove compactness. Indeed, a function $f(x) = x$ if x is real and 0 otherwise would be in $L^1(\mu)$ but is clearly not compact. Similarly, A_f would not necessarily be closed.

Problem 21 We claim that all completions are isomorphic.

For sake of contradiction, let Y_1, Y_2 be two unique completions of X. We define f_1, g_1, f_2, g_2 are bijective maps which map $X \to Y_1, Y_2$ and back respectively. These are defined as $f(y) = y \; \forall y \in Y$ and $f(x) = \lim f(y)$ for all $x = \lim y$. g is defined as the inverse, and it exists as our f is continuous on limits.

To show that these preserve the metric, note that we just need to check when $y_1, y_2 \in Y, y_1 \in Y y_2 \notin, y_1, y_2 \notin Y$ for Y_1 and Y_2. Through the definition of metrics, we can see that all of these cases preserve the metric. It follows that $f_2 \circ g_1^{-1} : Y_1 \to Y_2$ is an isomorphism and that all completions are isomorphic.

Problem 23 We let $\mu(X) = a$ and define $\mu' = \frac{\mu}{a}$. We notice that, since we scale down μ and $x^{\frac{n+1}{n}}$ is convex, we have Jensen's and

$$\left(\int_X |f|^n d\mu'\right)^{\frac{n+1}{n}} \leq \int_X |f|^{n+1} d\mu'$$

as $n \to \infty$, we note that $x^{\frac{n+1}{n}} \to x$, so equality becomes strict. Therefore, we can rearrange this inequality to have

$$\lim_{n\to\infty}\frac{\int_X |f|^{n+1} d\mu}{\int_X |f|^n d\mu} = ||f||_\infty$$

as desired. Note that this is because we have $\int_X f d\mu = a\int_X f d\mu'$ and our value of a, $a^{\frac{n+1}{n}} \to a$ as $n \to \infty$.

Problem 25 We again rewrite $\mu' = \frac{\mu}{a}$ where $\mu(E) = a$. We notice that, since log is concave, Jensen's is reversed and we have

$$\log\int_E f d\mu' \geq \int_E \log f d\mu' \implies \log(\frac{1}{a}) \geq \frac{1}{a}\int_E \log f d\mu$$
$$\implies \int_E \log f d\mu \leq \mu(E)\log\frac{1}{\mu(E)}$$

similarly, since x^p is concave for $0 < p < 1$, we find that

$$\left(\int_E f d\mu'\right)^p \geq \int_E f^p d\mu' \implies \frac{1}{a^p} \geq \frac{1}{a}\int_E f^p d\mu \implies \int_E f^p d\mu \leq \mu(E)^{1-p}$$