Math 6520 Differentiable Manifolds

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1 Aug 29

1.1 Introduction

Recall the chain rule: $u \xrightarrow{\varphi} v \xrightarrow{\psi} \mathbb{R}^p$ we have $D_u(\psi \circ \varphi) = D_{\varphi(u)}\psi \circ (D_u\varphi)$.

Def. *$\varphi : U \to \mathbb{R}^m, U \subset \mathbb{R}^m$ open is an **immersion** if φ is differentiable at all $u \in U$ and $T_u\varphi$ is $1:1$, where T_u is the derivative at u.*

Ex.

(1) $\mathbb{R} \to \mathbb{R}^2$ given by $\theta \to (\cos\theta, \sin\theta)$.

(2) Not: $(x, y) \to 0$.

(3) Open segment in $\mathbb{R}^1$ can be mapped to a circle with a open segment jutting out.

(4) Not: $x \to x^3$ because the derivative is not injective.

Def (Inductive Definition of Manifold Embedded in $\mathbb{R}^n$)**.**

(1) $\mathbb{R}^n$ is a n-manifold

(2) Open subsets of n-manifolds are submanifolds

(3) $M \subseteq \mathbb{R}^n$ (not necessarily open) and $\forall m \in M$, there exists a open $U \subseteq M, U \ni m$ (openness defined by subspace topology) and an open $V \subseteq \mathbb{R}^k$ then there exists an immersion

$$\varphi : V \to U \hookrightarrow \mathbb{R}^n$$

then M is a k-submanifold in $\mathbb{R}^n$. These are also manifolds.

Def. ***Chart**: (U, V, φ) on M around m.*

1.2 Various Flavors of Manifolds

Def. *Each φ is ∞ differentiable then it is a **smooth manifold**. If φ is continuous **topological manifolds**. Twice differentiable would be C^2 manifold.*

Ex. *Not: a closed interval of the real line.*

Def. *If $M \subseteq \mathbb{R}^n$ is a k-submanifold and (U, V, φ) is a chart around $m \in M$. We can reorient (through translation) s.t. $\varphi(0) = m$.*

$$T_m U := im(\eta_U \circ D_{\vec{0}} \varphi)$$

*and this is the **tangent space** at m where η_U is an inclusion mapping. Note that the dimension is k as φ is an immersion and also the tangent space is isomorphic with $\mathbb{R}^k$.*

Prop 1.1. *The tangent space $T_m M$ is independent of our choice of chart.*

Proof. Given two charts $(U_1, V_1, \varphi_1), (U_2, V_2, \varphi_2)$ note that $U_1 \cap U_2$ is a nbhd of m. Let $V_1' = \varphi_1^{-1}(U_1 \cap U_2)$ and $V_2' = \varphi_2^{-1}(U_1 \cap U_2)$. Note that by homeomorphism there exists a homeomorphism $b : V_1' \to V_2'$ s.t. $\varphi_1' = \varphi_2' \circ b$. Now we have that

$$D_{\vec{0}} \varphi_1' = D_{\vec{b(0)}} \varphi_2' \circ D_{\vec{0}} b = D_{\vec{0}} \varphi_2' \circ D_{\vec{0}} b$$

Note that that $D_{\vec{0}} \varphi_1'$ and $D_{\vec{0}} \varphi_2'$ are linear and 1:1, which means that $D_{\vec{0}} b$ is linear and 1:1, so it is onto, so $D_{\vec{0}} b(V_1') = V_2'$. Similarly, we can see that V_2' and V_1' are also bijective and it follows that φ_1' and φ_2' have the same image, which means that the tangent spaces are the same. □

Def. *Let M, N be manifolds. A function $\psi : M \to N$ is **smooth** (∞-diff) if there exists charts $(U_M, V_M, \varphi_M), (U_N, V_N, \varphi_N)$ s.t. $m \in U_M \subseteq M$ $\psi(m) \in U_N \subseteq N$ and $\varphi_N^{-1} \circ \psi \circ \varphi_M : V_M \to V_N$ is smooth from $\mathbb{R}^m \to \mathbb{R}^n$.*

Def. *We can define the **category** of **manifolds**. Objects are (smooth) manifolds, morphisms are smooth functions, and invertible morphisms are diffeomorphisms (homeomorphisms but infinitely differentiable both ways).*

Ex (Milner '60s)**.** *There exist 28 smooth manifolds that are homeomorphic to the 7-sphere $S^7 \subseteq \mathbb{R}^8$.*

2 September 3

2.1 Tangent Bundle

Def. *Let $M \subseteq \mathbb{R}^N$, then the **tangent bundle** of $TM = \{(x, y) : x \in M, y \in T_x M\}$. If $\dim(M) = k$, then the claim is that TM is a smooth manifold with* $\dim(TM) = 2k$.

Proof. It is $M \times \mathbb{R}^k$. Products of manifolds are manifolds with dimension $+2$. □

Theorem 2.1. *Let Mfld be the category of smooth manifolds. T : Mfld $\to$ Mfld is a functor.*

Proof.

- First, note that if M is a manifold, TM is a manifold.
- $f : M \to N, M, N \subseteq \mathbb{R}^k, Tf : TM \to TN$ then $Tf(x, y) = (f(x), Df_x(y))$.
- If I have a map from $f : N \to K, g : M \to N$, smooth manifolds, $T(f \circ g) = Tf \circ Tg$. The idea is that $D(f \circ g) = Df \circ Dg$.
- $T(\mathrm{id}_M) = \mathrm{id}_{TM}$.

So T is a functor from smooth manifolds to smooth manifolds. □

2.2 Inverse Function Theorem

Prop 2.1. *$f : X \to Y$ manifolds of same dimension, if $Df_x = T_xX \to T_{f(x)}T$ is an isomorphism, f is a local diffeomorphism at x.*

Proof.

We see that since Df_x is an isomorphism, then $\dim X = \dim Y$ and furthermore

$$\begin{tikzcd} X \arrow[r, "f"] & y \\ 0 \in U \arrow[u, "\varphi_1"] \arrow[r, "g"] & 0 \in V \arrow[u, "\varphi_2"'] \end{tikzcd}$$

Then we see that $Dg_0 : \mathbb{R}^k \to \mathbb{R}^k$ is a linear map and in particular has an inverse (by IFT for Euclidean open subsets) if $\det Dg_0 \neq 0$. But this only happens in particular when it is an isomorphism. □

Remark. *$f : \mathbb{R}^1 \to S^1$ by $x \to e^{ix}$, we see that this is a local diffeomorphism but not global.*

2.3 Immersions

Def. *$f : X \to Y$ manifolds, f is an **immersion** at x if Df_x is injective.*

Ex. *$f : \mathbb{R}^k \to \mathbb{R}^l$ by $(x_1, x_2, \ldots, x_k) \to (x_1, \ldots, x_k, 0, 0, \ldots 0)$. This is the **canonical immersion**.*

Theorem 2.2. *$f : X \to Y$ is an immersion at x. Then after some local parametrization around x, f is a canonical immersion.*

Proof.

$$\begin{tikzcd} X \arrow[r, "f"] & y \\ 0 \in U \arrow[u, "\varphi_1"] \arrow[r, "g"] & 0 \in V \arrow[u, "\varphi_2"'] \end{tikzcd}$$

For $f : X \to Y$ immersion, we want to find φ_1, φ_2 s.t. $g : \mathbb{R}^k \to \mathbb{R}^l$, since g is an immersion at 0 (local parametrization is also immersion), $Dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ and has rank k.

WTS after some change of coordinates, then g is canonical immersion. WLOG, $Dg_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$ with change of basis in $\mathbb{R}^l$. Define $G : U \times \mathbb{R}^{l-k} \to \mathbb{R}^l, G(x, z) = g(x)+(0, z)$. $g = G \circ c$, where c and this is the canonical immersion $x \to (x, 0)$. In particular $DG_0 = I_l$ by IFT, G is a local diffeomorphism, so we can parametrize around y with $\psi \circ G$. Shrinking U and V to $\tilde{U}, \tilde{V}$ we have

$$\begin{tikzcd} X \arrow[r, "f"] & y \\ 0 \in U \arrow[u, "\varphi_1"] \arrow[r, "c"] & 0 \in V \arrow[u, "\varphi_2 \circ G"'] \end{tikzcd}$$

and this is our desired canonical immersion. $\square$

Remark. *Image of immersion may not be a submanifold. For example, $f : \mathbb{R}^1 \to$ a figure 8 and this is not injective at the center.*

Another example is $f : \mathbb{R}^1 \to$ a figure 8 where we curve around the $\pm\infty$ converge to the intersection point (like an S). This is injective and onto but it is not a manifold since the preimage of a nbhd around the intersection point has infinitely many points.

Def. *$f : X \to Y$ continuous from topological spaces X, Y. we say f is **proper** if $f^{-1}(K)$ is compact in X for compact K.*

Ex. *Let $f : \mathbb{R}^1 \to \mathbb{R}^1/\mathbb{Z} \times \mathbb{R}^1/\mathbb{Z}, x \to (x, ax)$ (torus) where $a \notin \mathbb{Q}$ where we mod by 1. The preimage of a point in this has infinitely many, so not proper.*

Def. *$f : X \to Y$ manifolds. f is an **embedding** if $f : X \to f(X)$ is a diffeomorphism. Therefore, $f(x)$ is a submanifold of Y.*

Theorem 2.3. *$f : X \to Y$ is an immersion. If f is injective and proper, f is an embedding.*

Proof.

- $f : X \to f(X)$ is an open map. For $x \in X$ and nbhd U, then want to show that $f(U)$ is open in $f(X)$. Suppose $f(U)$ is not open, then there exists $(y_n) \subseteq f(X)$ s.t. $y_n \notin f(U)$ but $y_n \to y \in f(U)$. $\exists! x$ s.t. $f(x) = y$ and $\exists! x_n$ s.t. $f(x_n) = y_n$. Consider $\{x, x_1, x_2, \dots\}$. Note that $f^{-1}(\{y, y_1, \dots\})$ which is compact, so (x_n) has a convergent subsequence $x_{n_i} \to x'$ which means that $f(x_{n_i}) = y_{n_i} \to f(x') = y = f(x)$ and since it is injective $x = x'$, which implies that $f(X_{n_k}) = y_{n_k} \in f(U)$ for some n_k, which is our contradiction.

- Given $x \in X$, using local immersion Theorem, f is local diffeomorphism. By IFT, there exists $f^{-1} : f(x) \to x$ that is smooth around x.

□

3 Sep 5

3.1 Submersions

Def. *$f : X \to Y$ smooth map, manifolds, f is a **submersion** at x if $Df_x : T_xX \to T_yY$ is onto where $y = f(x)$.*

Ex. *$f : \mathbb{R}^k \to \mathbb{R}^\ell, k > \ell$ where $f(x_1, \dots, x_\ell, \dots, x_k) = (x_1, x_2 \dots, x_\ell)$. This is the **canonical submersion**.*

Theorem 3.1. *$f : X \to Y$ smooth map between manifolds, f submersion at x, then $\exists$ local parametrization s.t. f is canonical submersion.*

Proof.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ 0 \in U \subseteq \mathbb{R}^k \arrow[u, "\varphi_1"] \arrow[r, "g"] & 0 \in V \subseteq \mathbb{R}^\ell \arrow[u, "\varphi_2"] \end{tikzcd}$$

where $\varphi_1(0) = x, \varphi_2(0) = y = f(x)$ and $k > \ell$. $Dg_0 : T_0U \to T_0V$ has rank ℓ. We can rewrite Dg_0 as $\begin{bmatrix} I_\ell \mid 0 \end{bmatrix}$ after a change of basis of $\mathbb{R}^k$.

Introduce $G : U \to \mathbb{R}^k = \mathbb{R}^\ell \times \mathbb{R}^{k-\ell}$. $G(a) = (g(a), a_{\ell+1}, \dots, a_k)$. Can see that $g = (\text{canonical submersion}) \circ G$ and $DG_0 = I_k$, so G is a local diffeomorphism by IFT. If we denote the canonical submersion as s, we have

$$\begin{tikzcd} x \arrow[rr, "f"] & & y \\ u \arrow[u, "\varphi_1"] \arrow[r, "G"] & G(u) \arrow[r, "s"] & V \arrow[u, "\varphi_2"'] \end{tikzcd}$$

and setting $\varphi_3 = \varphi_1 \circ G^{-1}$ on an open set $\tilde{U}$ we have

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \tilde{U} \arrow[u, "\varphi_3"] \arrow[r, "s"] & \tilde{V} \arrow[u, "\varphi_1"'] \end{tikzcd}$$

□

Def. *$f : X \to Y$ smooth map between manifolds. We say $y \in Y$ is a **regular value** if $\forall x \in f^{-1}(y)$, f is a submersion at x.*

Def. *y is a **critical value** if y is not regular.*

Theorem 3.2 (Preimage Thm)**.** *$f : X \to Y$. If y is a regular value, then $L = \{x \in X : f(x) = y\}$ is a submanifold of X. $X \subseteq \mathbb{R}^k, Y \subseteq \mathbb{R}^\ell$.*

Proof. WTS $\forall x \in L$, $\exists U \subseteq X$ s.t. $U \cap L \xrightarrow{\sim} V \subseteq \mathbb{R}^{\dim L}$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ 0 \in U \arrow[u, "\varphi_1"] \arrow[r, "g"] & 0 \in V \arrow[u, "\varphi_2"'] \end{tikzcd}$$

where $\varphi_1(0) = x, \varphi_2(0) = f(x) = y$, $g(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$. $\varphi_1^{-1}(L) \cap U = g^{-1}(0) \cap U$, $g^{-1}(0) = \{(0, \ldots, 0, x_{k+1}, \ldots, x_\ell)\} \cong \mathbb{R}^{k-\ell}$.

L is a submanifold with dimension $k - \ell = \dim X - \dim Y$. □

Corrollary. *$Z = f^{-1}(y)$, $f : X \to Y$ y regular, then $\forall x \in Z, T_xZ =$ $\ker(Df_x)$.*

Proof. Since $Z = f^{-1}(y), T_xZ \subseteq \ker(Df_x)$. By dim condition $T_xZ = \ker(Df_x)$. □

Ex. *$\Xi = \{(x, y, z) \in \mathbb{R}^3, x^3 + y^3 + z^3 + 3xyz = 1\}$ is a smooth manifold. If $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^3 + y^3 + z^3 + 3xyz$, then $\nabla f = (3x^2 + 3xyz, 3y^2 + 3xz, 3z^2 + 3xy)$. If $\nabla f = 0$, then we need that all are 0, but this is not inside the set. Therefore Ξ is a smooth manifold of $\dim = 3 - 1 = 2$.*

Ex. *$n \in \mathbb{N}, n \geq 2$, $O(n) = \{A \in M(n), AA^T = I_n\}$. WTS that $O(n)$ is a smooth manifold of $M(n)$. Define $f : M(n) \to S(n)$ by $f(A) = AA^T$. We see that $O(n) = f^{-1}(I_n)$ and WTS that I_n is a regular value of f, or that $Df_A : T_A M(N) \to T_f(A)S(n)$ is surjective. We calculate*

$$\begin{aligned} Df_A(B) &= \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s} \\ &= \lim_{s \to 0} \frac{(A + sb)(A + sB)^T - AA^T}{s} \\ &= BA^T + AB^T \end{aligned}$$

Now we need to show Df_A is onto. Note $C \in T_{f(A)}s(n) = s(n)$ and $T_A M(n) = M(n)$ since they are Euclidean. $C = \frac{C + C^T}{2}$ and want $BA^T = \frac{C}{2}, B = \frac{CA}{2} \implies Df_A$ is onto. So A is a regular value. By Preimage Thm, $O(n)$ is a submanifold of $M(n)$ with $\dim O(n) = \dim M(n) - \dim S(n) = \frac{n(n-1)}{2}$.

Remark. *M manifold cut by $g_1, g_2, \ldots, g_r, g_i : M \to \mathbb{R}$. We say $g_1, \ldots, g_r$ are independent functions on M around $x \in M$ if $(Dg_i)_x$ are independent vectors of $T_x M$.*

Theorem 3.3. *If y is a regular value $f : X \to Y, \operatorname{codim} f^{-1}(y) = \ell$. Then $f^{-1}(y)$ is actually a cut by ℓ functions $g_1, g_2, \ldots, g_\ell : X \to \mathbb{R}$ with $f^{-1}(y) = \bigcap_{i=1}^{\ell} g_i^{-1}(0)$*

Proof. Pick $h : V \ni y \xrightarrow{\sim} \tilde{V} \subseteq \mathbb{R}^\ell, y \to \vec{0}. f^{-1}(y) = (h \circ f^{-1})(0) = \bigcap_{i=1}^{\ell} g_i^{-1}(0)$ where $h \circ f = (g_1, g_2, \ldots, g_\ell)$. $\square$

3.2 Transversality

We have shown that $f^{-1}(y)$ is a submanifold when y is regular. If Z is a submanifold of Y, what condition make $f^{-1}(Z)$ a submanifold of X.

Def. *$f : X \to Y, Z \subseteq Y$ is a submanifold. f is **transversal** to Z $f \pitchfork T$ if $\operatorname{im}(Df_x) + T_y Z = T_y Y$ for all $x \in f^{-1}(Z)$.*

Theorem 3.4. *If $f \pitchfork Z$ then $f^{-1}(Z)$ is a submanifold.*

Proof. Ask Knutson. $\square$

Ex. *$X, Y \subseteq Z, i : X \to Z$ inclusion map. We say $X \pitchfork Y$ if $i \pitchfork Y$.*

4 September 10

4.1 Homotopies and stable/open properties

Def. *A continuous function $f : X \times [0,1] \to Y$ is a* ***homotopy*** *between f_0 and f_1 where $f_t(x) = f(x, y)$.*

Def. *$g, h : X \to Y$ are* ***homotopic*** *if there exists a homotopy s.t. $f_0 = g, f_1 = h$.*

Ex.

- *More generally, we can consider a family of maps $f_s : X \to Y$ given by a $f : X \times S \to Y$ by $f(x, s) = f_s(x)$.*
- *Non example: if X is one point, Y two points and $g(x) \neq h(x)$ then no homotopy between the two.*

Def. *A property p of maps is called* ***stable*** *or* ***open*** *in families if*

1. *For all homotopic $f : X \times I \to Y$ if f_0 is p, then f_t is p for all small enough t.*
2. *For all families over S of maps $f : X \times S \to Y$ $\{s \in S | f_s \text{ is } p\}$ is open in S.*

We can see that 2 implies 1 as we take $S = I$ and $s = 0$. For 1 to 2, we take a path $P \subset S$ and parametrize $f : I \to P$ and take $s = f(0)$.

Theorem 4.1. *The following p are stable properties of maps from compact $X \to Y$.*

(a) *Local diffeomorphisms (including immersions and submersions).*

(b) *Immersions*

(c) *Submersions.*

(d) *Transverse maps to some $Z \subseteq Y$.*

(e) *Embeddings (injective immersions)*

(f) *Diffeomorphisms*

Ex. *Non example: $\mathbb{R} \to \mathbb{R}^2$ by $x \to x + t$.*

Def (transverse)**.** *For $f : W \to M$, f is* ***transverse*** *to z if $\forall\ m \in M$ s.t. $f(w) = z$, $T_z Z + \operatorname{im}(T_w f) = T_z M$*

Proof. Ask knutson about this ☐

5 September 12

5.1 Critical and Regular values

Recall that $f : M \to N$ smooth map between manifolds and $T_pf : T_pM \to T_{f(p)}N$.

Def. *p is a **regular point** if T_pf is onto and is a **critical point** if it is not. The regular condition is an open coordination, and being critical is a closed condition (since determinants are 0 for closed and open is non zero)*

Ex. $A_1(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$. *Everything on the positive is regular, but non-positive are critical.*

What about $\{f(p)\}$.

Def. *$q \in N$ is a **critical value** if $\exists$ a critical point $p : f(p) = q$ and a **regular value** if there doesn't exist such a p.*

Theorem 5.1. *If q is a regular value, then $f^{-1}(g) \subseteq M$ is a submanifold of codimension* $\dim N$.

Ex (can have manifolds not preimages)**.** *Consider a Mobius strip M^2 in $\mathbb{R}^3$. Can't get $M^2 = f^{-1}(0)$ a regular value $f : U \to \mathbb{R}$. Where U is an open set $M \subseteq U^3 \subseteq \mathbb{R}^3$ since U has a positive and negative side but the Mobius strip does not.*

Ex (critical values might not be closed set)**.** $A_2(x) = A_1(x)A_1(1-x)$ *is a bump function supported in* $[0, 1]$ *and smooth. If* $f(x) = \sum_{i=1}^{\infty} A_2(x - 2i) * (\frac{1}{2} + 2^{-i})$*. Critical values are* $A_2(0) * (1/2 + 2^{-i})$

Theorem 5.2 (Sard)**.** *The set of critical values is measure* 0 *in N.*

Def. *$X \subseteq \mathbb{R}^k$ is **measure zero** if for all ϵ there is a countable open cover s.t. the volume of the balls is less than ϵ. This definition is invariant under diffeomorphism from $\mathbb{R}^k \to \mathbb{R}^k$.*

Corrollary. *The set of of regular values of $f : M \to N$ is dense. Note that this uses the fact that f is smooth, unlike "space-filling curves".*

Remark. *Sard's theorem has a counterpart in algebraic geometry called "generic freeness" but only holds over fields of characteristic* 0.

Ex. *$f : \mathbb{C} \to \mathbb{C}$, $f(x) = x^2$, $f^{-1}(t)$ is 2 pts for $t \neq 0$, 1 pt for $t = 0$.*

Ex. *$\mathbb{F}_2 \to \mathbb{F}_2$ has $f^{-1}(t)$ is 1 point.*

Prop 5.1. *If $\{f_i : M_i \to N\}$ is a countable sequence, then $\bigcap_{i=1}^{\infty}$(regular values of f_i) is dense.*

Proof. Union of measure 0 sets is measure 0, and their complement is our desired. □

5.2 Linear Algebra Around Critical Points

Def. *Let p be a critical point of $f : M \to \mathbb{R}$. In coordinates on M near p, the second derivatives give a symmetric $n \times n$ matrix, the* ***Hessian***. *Without picking coordinates we can say its defining a symmetric bilinear form on T_pM.*

In one set of coordinates, get H matrix and can use change of coordinates $H \to AHA^T$. A is the derivative of the map from one open chart's open set to another's.

Theorem 5.3 (Sylvester's law of inertia). *For all symmetric H, exists an invertible A s.t. AHA^T is a diagonal matrix with several $+1, -1, 0$ in that order and the number of $+1, -1, 0$ is invariant.*

Prop 5.2. *If there are no 0 ie H is invertible then the number of 1 and -1 is stable in H. Ie if we perturb H by a small symmetric perturbation then the numbers remain the same.*

Ex. *$M = s^2$ and map using the height. There are 2 critical values.*

Ex. *M is torus T^2 (stood up on it's side) and use height function. There are 4 critical values at the top and bottom of the circles. The bottom point has two $+1$ in it's hessian, as it looks like $x^2 + y^2$ and the second derivative is both positive. The middle two points are $+-$, and the top point is $--$.*

Ex. *T^2 but laid on it's side. There are two critical values corresponding to circles. The bottom circle is $+0$ and the top is -0 since is is flat on the x but increasing/decreasing on the y.*

5.3 Some Morse Theory

Def. *A critical point $p \in M \to \mathbb{R}$ in f is* ***Morse*** *if the Hessian at p is "nondegenerate" (invertible, only +1, -1).*

Def. *A function $f : M \to \mathbb{R}$ is a* ***Morse function*** *if all critical points are Morse.*

Ex. *In our two torus examples, the top is Morse but the bottom is not Morse. Note that in the top example in the z-axis there is one symmetry (a flip) while in the second the flat torus has infinitely many symmetries (rotation in z-axis).*

Lemma (Morse Lemma)**.** *If p is a Morse crit point of $f : M \to \mathbb{R}$ then $\exists$ local coordinates $c : U \to M$ taking $0 \to p$, U open in $\mathbb{R}^k$ s.t. $U \xrightarrow{c} M \xrightarrow{f} \mathbb{R}^k$ then $(f \circ c)(x_1, \dots, x_k) = \sum_{i=1}^{r_+} x_i^2 - \sum_{j=r_+ +1}^{k} x_j^2$*

6 September 12

6.1 Recall

Let $f : M \to \mathbb{R}$ a smooth function. A point $p \in M$ is critical if $T_p f = 0$ (not onto since dimension of $\mathbb{R}$ is 0). A critical point p is is Morse if the hessian at p is nondegenerate, $T_pM \times T_pM \to \mathbb{R}$.

6.2 Morse Theorem

Def. *Let $U \hookrightarrow M \subseteq \mathbb{R}^n$ where M is k-dimensional be open. Cal u* ***adapted to coordinates*** *if $\exists k$-subset $S \subseteq \{1, \dots, n\}$ so $U \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{R}^S$ is diffeomorphic to its image.*

Ex. *On the circle, if u_1 is an arc in quadrant I it is diffeomorphic to the projection on the x-axis and y axis (adapted to 1 or 2). If u_2 is an arc that intersects $(1, 0)$, it is diffeomorphic to the projection on x-axis (adapted to 2). If u_3 is a curve that intersects $(0, 1), (0, -1)$ then it is not adapted to either.*

Remark. *Note that this means that we can create the chart at $\mathbb{R}^s$ and get the chart to $\mathbb{R}^n$. In 2230-2240 we can define a manifold through this process through out system of charts.*

Def. *$V^\star$ is the linear maps from V to $\mathbb{R}$.*

Theorem 6.1. *Let $M^k \hookrightarrow \mathbb{R}^n$. Let $f_0 : M^k \to \mathbb{R}$. Define $f : M \times (\mathbb{R}^n)^\star \to \mathbb{R}$ by $(m, \vec{a} \to f_0(m) + \left\langle \vec{a}, m \right\rangle)$ (note we can just use $\mathbb{R}^n$ and use dot product). For a.e. $\vec{a}$, $f_{\vec{a}} = f(m, \vec{a}) : M \to \mathbb{R}$ is Morse.*

Remark. *This lets us define things about M by using Morse functions since they exist.*

Def. *If $f : M \to \mathbb{R}$ is Morse, M compact. Let $\chi(M, f)$ be the sum of the determinants Sylvester normal form of the Hessian of f at p for all critical points f at p. Note that this sum is finite by compactness. This is the **Euler Characteristic** of M*

Ex. *If we consider a sphere and take projection to z-axis. The Euler characteristic is 2 as the two critical points are the top with SNF $[-1, -1]$ and the bottom with SNF $[+1, +1]$.*

If the indent the top point by pushing it down, then we have 4 critical points $[-1, -1], [-1, -1]$ for the two newly created points, the top points with $[+1, -1]$ and the bottom points is $[+1, +1]$. The Euler Characteristic is 2 as well.

$$insertdiagramshere$$

Proof. **Part 1** If $M = U \hookrightarrow \mathbb{R}^k$ is open set. Define $g_{\vec{a}} : U \to (\mathbb{R}^k)^\star$ essentially as $D(f_{\vec{a}})$. Note that $Df_{\vec{a}}$ is a linear map from $U \to \mathbb{R}$ as $f_{\vec{a}} : U \to \mathbb{R}$. $g_{\vec{a}} = Df_{\vec{a}}$ in this case.

$f_{\vec{a}}(m) = f_0(m) + \left\langle \vec{a}, m \right\rangle$ so $g_{\vec{a}} = Df_{\vec{a}} = Df_0 + \vec{a},$. Note that $\vec{a} \in (\mathbb{R}^k)^\star$.

1. If $f_{\vec{a}}$ is critical at p then $g(p) = -\vec{a}$, where $g(p) = T_p f_0$.
2. Hessian of $f_{\vec{a}} = Dg_{\vec{a}}$, where $T_p g_{\vec{a}} : \mathbb{R}^k \to (\mathbb{R}^k)^\star$ as th

If $-\vec{a}$ is a regular value of $g : U \to (\mathbb{R}^k)^\star$ (using Sard) and $f_{\vec{a}}$ is critical value at p, then this occurs iff $T_p g$ is onto, which means that $T_p g$ is invertible so the Hessian is nondegenerate and f is Morse at p.

Part 2 Cover M with open sets U^k adapted coordinates. Since it is second countable, we only need countably many. Assume some particular U is adapted to $\{1, 2, \ldots, k\}$. For each $\vec{c} \in \mathbb{R}^{n-k}$ apply part 1 to $f_{(o^k, \vec{c})}$ to find a.e. $\vec{b} \in \mathbb{R}^k$ has $f_{(\vec{b}, \vec{c})}$. If we say that this $\vec{b}$ is $S_{\vec{c}} \subseteq \mathbb{R}^k$ where $S_{\vec{c}}$ is of measure 0. Then

$$\int_{\vec{c} \in \mathbb{R}^{n-k}} 1_{\vec{c}} = 1_{\bigcup_{U_k} \{\vec{b}, \vec{c}, \vec{b} \in U\}}$$

And this is the characteristic of a set of measure 0. $\square$

Remark. ***Note that $Df_a(p) = T_p f_a$ is a linear functional on $\mathbb{R}^k$. Knutson substitutes T instead of D but I'll use T since it's easier to parse.***

Ex. *Under our Morse function projection to height axis, $\chi(T^2) = 0$. $\chi(\{0\}) = 1$. $\chi(S^1) = 0$ since bottom is $+$ top is $-$. $\chi(\text{genus}) = 2 - 2g$ where the genus is a bunch of connected torus's (g connected). Euler characteristic of Klein bottle is 0. The real project plane $S^2/\pm$ has characteristic 1.*

7 September 19

7.1 Whitney Embedding Theorem

Remark. *We have shown that surfaces like the Klein bottle and Boy surface can be immersed in $\mathbb{R}^3$ but not embedded. The question is for $M^k \hookrightarrow \mathbb{R}^n$ an embedding what is good value of n (project to linear space).*

Theorem 7.1 (Whitney embedding)**.** *A.e. linear projection $\mathbb{R}^n \to \mathbb{R}^{2k+1}$ gives an embedding $M^k \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{R}^{2k+1}$ when M^k is compact.*

Proof. It is enough to project $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n-1}$ when $n > 2k+1$ and then induct. Recall that if $M \hookrightarrow \mathbb{R}^n$ then the tangent bundle $TM \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$, $TM = \{(m, \vec{v} : \vec{v} \in T_m M)\}$ is a $2k$ submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Take $h : M \times M \times \mathbb{R} \to \mathbb{R}^n$ by $(x, y, t) \to t(f(x) - f(y))$ with $g : TM \to T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \overset{\pi_2}{\twoheadrightarrow} \mathbb{R}^n$ which is $(m, \vec{v}) \to Tf(\vec{v})$. Let $\vec{a} \in \mathbb{R}^n$ be in neither image by Sard. Now claim $M \overset{f}{\hookrightarrow} \mathbb{R}^n \overset{orthogonal}{\twoheadrightarrow} \vec{a}^{\perp} := \{\vec{b} \in \mathbb{R}^n : \vec{b} \perp \vec{a}\}$ is an embedding. Note that the kernel of this orthogonal map π is $\mathbb{R}\vec{a}$.

- Injectivity:If $(\pi \circ f)(x) = (\pi \circ f)(y)$, then this occurs iff $f(x) - f(y) = t\vec{a}$ for some $t \in R$ not 0. This implies that $h(x, y, , \frac{1}{t}) = \vec{a}$, which is a contradiction.
- If $T(\pi \circ f) = (T\pi \circ Tf)(\vec{v}) = \vec{0}$. Note that since π is linear it's derivative is itself, so we have that $\pi(T_x f(\vec{v})) = \vec{0}$ which means that $Tf(\vec{v}) \in \mathbb{R}^{\vec{a}}$ so it is $t\vec{a}$ for some $t \neq 0$. Now $g(x, \frac{1}{t}) = \vec{a}$.

$\square$

Remark. *An **Abstract k manifold** M^k not in any $\mathbb{R}^n$ is a second countable Hausdorff space where every point is locally homeomorphic to $\mathbb{R}^k$. In* *particular it has an atlas $\{\varphi_i : U_i \hookrightarrow M\}$ s.t. φ_i^{-1} that $\varphi_i(U_i) \cap \varphi_j(U_j)$ is open in U_j and U_i. This means it is* ***countable****. There exists a Whitney Embedding Theorem for these (can find in Munkres I think).*

Ex. *Non example: The long line is multiple $[0, 1]$ glued together, one for each ordinal $n < \omega_1$ which is the uncountable ordinal.*

7.2 Manifolds with Boundary

Def. *A manifold with boundary $M^k \subseteq \mathbb{R}^n$ has charts modeled on open sets in $\mathbb{R}^{k-1} \times \mathbb{R}_{\geq 0}$.*

Ex. *An example is in the ball B^n for $|x| \leq 1$. In the middle it is R^k and at the boundary it is $\mathbb{R}^{k-1} \times \mathbb{R}_{\geq 0}$ (open set of $\mathbb{R}^k$ intersected with $\mathbb{R}_{\geq 0}$).*

Remark. *A technical issue at the boundary is that at the boundary the half space demands an extension.*

Theorem 7.2. *If $(M^k, \partial M)$ is a manifold with boundary, then ∂M is a $(k-1)$ manifold without boundary.*

Corrollary. *$\partial^2 = 0$.*

Remark. *Note that $\partial(M \times N) = (M \times \partial N) \cup (\partial M \times N)$, M, N manifolds with boundary. This is like Leibniz's rule.*

Theorem 7.3. *If M is a compact connected 1-manifold with boundary (note that it might have empty boundary). Then $M \cong [0, 1]$ or S^1.*

Proof. Perturb the constant function $f = 0$ to a (linear) Morse function$f : M \to \mathbb{R}$. Make a graph w/ vertices are the connected components of regular points. Then edges correspond to connections on critical points. These give points of degree 1 and degree 2. We need to prove that M connected implies that the graph Γ is connected. $\square$

Corrollary. *If M is compact 1 manifold with boundary, then there are two divides the total number of points in ∂M.*

Def. *A* ***retraction*** *of M to $N \subseteq M$ a submanifold, is a map $r : M \to N$ where $r|_N = \mathrm{id}_N$.*

Theorem 7.4. *If M compact with boundary, then there doesn't exit a retraction $r : M \to \partial M$.*

Proof. Let n be a regular value of r. If n is a regular value of r, then $r^{-1}(n)$ is a 1-manifold with boundary (dimension is same as codimension of ∂M) that is also compact. $\partial r^{-1}(n) = \partial M \cap r^{-1}(n) = \{n\}$, where we invoke sard. $\square$

Corrollary (Brouwer)**.** *If $f : B^n \to B^n$ is smooth then f has a fixed point.*

Proof. If not we construct a retraction. But those don't exist. The retraction is $r(x) = f(x) \to x \to \partial B^n$ where we find the line containing $f(x)$ and x and map to the boundary starting from $f(x)$. $\square$

8 September 24

8.1 Manifolds w/Boundary Revisited

Theorem 8.1 (Sard's Thm for Manifold with Boundary)**.** *Let $(M, \partial M) \xrightarrow{f} Y$. Then a.e. $y \in Y$ is a regular value for $f\mid_{M \setminus \partial M}$ and $\partial f = f\mid_{\partial M}$ and regular for the open $M_+ \supseteq M \cup \partial M$ which we use to define the region for f being smooth. (IE the extension of M to another manifold). If $y \in Y$ is a regular value, then $f^{-1}(y)$ is a submanifold with boundary and $\partial f^{-1}(y) = f^{-1}(y) \cap \partial M$.*

8.2 Transversality is stable

Theorem 8.2. *The $F : X \times S \to Y$ where X is a manifold with boundary, S manifold, Y manifold and $Z \hookrightarrow Y$ a submanifold s.t. F is transverse to Z. If F, ∂F are transverse to z, then a.e. $s \in S, F_x : X \to Y$ and ∂F_x are transverse to z.*

Proof. Pick s a regular value of $F^{-1}(z) = W \to X \times S \to S$. We want if $F_s(x) = z \in Z \subseteq Y$ that $T_{F_s}T_x(X) + T_zZ = T_zY$. We know $T_F(T_{x,s}(x \times s)) + T_zZ = T_zY$. Then we see that $F_x(s,) = F_s(x) = z$ so $T_{(x,z)}W \overset{T\pi}{\twoheadrightarrow} T_s(S)$ taking $(\vec{u}, \vec{e}) \to \vec{e}$. Then $TF_{(x,s)}((\vec{w}, \vec{e}) - (\vec{u}, \vec{e})) - \vec{a} \in T_zZ$ since $T_{(x,s)}F(\vec{u}, \vec{e}) \in T_zZ$. $\square$

Remark. *There are two more technical results.*

1. *Let $f : X \to Y \supseteq Z$ and $\partial Y = \partial Z = \emptyset$. Then there exists perturbations of f (homotopies F w/ $F_0 = f$) to make f transverse to Z.*

2. *If f is already transversal on ∂X then we can perturb f (to $\pitchfork z$) without changing ∂f.*

Def. *If $f : X \to Y \supseteq Z$ and $\dim X + \dim Z = \dim Y$ and f transverse to Z. Note that an equivalent of transversality is $\operatorname{codim}(X \cap Z) = \dim Y - \dim(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z$. Let $I_2(f, z)$ be the number of points in $f^{-1}(z)$ mod 2.*

Theorem 8.3. *Let F be a homotopy from f_0 to f_1. ie $F : X \times [0, 1] \to Y$. $f_0, f_1 \pitchfork Z \hookrightarrow Y$, X, Y, Z compact. Then $I_2(f_0, z) = I_2(f_1, z)$.*

Proof. Perturb F without changing f_0, f_1 so that it is transverse to Z. Now $F^{-1}(Z)$ is a 1 manifold with boundary $\subseteq X \times [0, 1]$. $\partial F^{-1}(z) = F^{-1}(z) \cap \partial(X \times [0, 1]) = F^{-1}(z) \cap (X \times \{0, 1\})$. Then $0 \equiv \#\partial F^{-1}(z) = \#f_0^{-1}(z) + \#f_1^{-1}(z)$. $\square$

Ex. *Let $X \hookrightarrow T^2 \hookleftarrow Z$.*

Def. *If $X \hookrightarrow Y \hookrightarrow \mathbb{R}^n$ and $TX \hookrightarrow TY \hookrightarrow Y \times \mathbb{R}^n$ submanifold. The **normal bundle** $N_X Y = \{(y, \vec{v}) \in TY : y \in X, \vec{v} \perp T_y X\}$.*

Remark. *If there exists a manifold $T^\star M$, the **cotangent bundle** to M is $T^\star M = \{(m, \vec{v}), m \in M, \vec{v} \in (T_m M)^\star\}$ which is a real morphism from $T_m M$ to $\mathbb{R}$.*

9 September 26

9.1 Connected

Remark. *Recall our two versions of "connected". 1 is that every continuous function $X \to \{\pm 1\}$ has exactly one value and version 2 is that for all points exists a continuous function $[0, 1] \to X$ that maps $0 \to x, 1 \to y$. Note that this is **path connected** condition.*

Def (One-connected). *M manifold and connected is **simply connected** if (equivalently)*

1. *for all x, y and $\gamma : [0, 1] \to M$ with $0 \to x, 1 \to y$ and γ is unique up to homotopy relative to γ.*
2. *If $\gamma : S^1 \to M$ is a curve then there exists an extension to D^2.*

Ex. *The sphere any two points are simply connected (and path connected). On the empty torus, any two points around a circle in the torus are not simply connected (since they are not homotopic since you can't go between them as it is hollow.)*

Theorem 9.1. *If $\gamma : S^1 \to M$ and Z codimension 1 in M submanifold with no boundary and $I_2(\gamma, Z) \neq 0$, then M is not simply connected.*

Proof. The contrapositive is that if M is simply connected we can use homotopy to take γ to a point, missing Z. $\square$

9.2 Transversality

Theorem 9.2 (ϵ-nbhd thm)**.** *If $\gamma \subseteq \mathbb{R}^m$, γ compact, then $\exists \epsilon > 0$ s.t. $\gamma^\epsilon := \{m \in \mathbb{R}^m, d(m, Y) < \epsilon\}$ has a map $\gamma^\epsilon \to \gamma$ to the nearest point that is well defined, smooth, and submersion.*

Proof. Recall the normal bundle $N_Y\mathbb{R}^m = \{(y, \vec{v}), \vec{v} \perp T_yY\}$. Map y to $y + \vec{v}$ with h. h is regular along $Y \hookrightarrow N_y\mathbb{R}^m$ so Th is an isomorphism. h is injective on $Y \hookrightarrow N_y\mathbb{R}^m$. This implies h is a local diffeomorphism in some U open $\supseteq Y$. For each $\epsilon > 0$ let $v_\epsilon := \{y \in Y : U \supseteq B_\epsilon(y)\}$ and $Y = \bigcup_{\epsilon > 0} V_\epsilon$ and since Y is compact we can find an finite cover $V_{\epsilon_1} \cup \cdots \cup V_{\epsilon_k}$ and $\epsilon = \min_k \epsilon_k$. $\square$

Corrollary. *If $f : X \to Y$ is a map and Z is a submanifold of Y with $\partial Z = \emptyset$, then f is homotopic to $g : x \to y$ s.t. $g \pitchfork Z, \partial g \pitchfork Z$.*

Proof. Let S be open in $\mathbb{R}^m$ around 0, $X \times S \to Y$ given by $(x, z) \to f(x) + s$ to the nearest point in Y if s is small enough. This composite is a submersion since we can vary S to give us the perturbations of Y (definition of submersion). For a.e. $s \in S$ $F_s \pitchfork Z, \partial F_s \pitchfork Z$ and as shown last time this implies $F_0 = f$. $\square$

Remark. *A better version is that if $W \hookrightarrow X$ is closed, $f\mid_W \pitchfork Z$ and $\partial f\mid_W \pitchfork Z$ then can insist that $g = f$ on W.*

Theorem 9.3 (Jordan curve theorem)**.** *Let M be connected and simply connected and $H \subseteq M$ closed with codimension 1 and $\partial H \neq \emptyset$. Then $M \setminus H$ has ≥ 2 components each of which has H on the boundary of closures.*

Proof. We need a function $s : M \setminus H \to [0, 1]$ that is continuous and onto. Pick $m \in M \setminus H$. For any $n \in M \setminus H$, pick $\gamma : [0, 1] \to M$ with $0 \to m, 1 \to n$ using the fact that M is connected. Let $s = I_2(\gamma, H)$ and we can perturb γ to be transverse to H (γ has dimension 1). Note that this is well defined since M is simply connected. $\square$

9.3 Orientation

Def. *An* ***orientation*** *of a finite dimensional real vector space V is a function $\mathcal{O}$*

$$\mathcal{O} : \{\textit{ordered bases of } V\} \to \{\pm 1\}$$

s.t. if $T : V \to V$ invertible and $[b_1, \dots, b_n]$ a basis then $\mathcal{O}(T(b_1) \dots T(b_n)) = \operatorname{sign}(\det T)\mathcal{O}(b_1, \dots, b_n)$.

Remark. *If $V = \{\vec{0}\}$. Then the ordered bases are $\{\emptyset\}$.*

Given V, W vector spaces and $V \oplus W := V \times W$ with coordinate wise vector space operations. Call orientations $\mathcal{O}_V, \mathcal{O}_W, \mathcal{O}_{V\oplus W}$ ***compatible*** *if for bases b_i of V, c_j of w then*

$$\mathcal{O}_V(b)\mathcal{O}_W(c) = \mathcal{O}_{V\oplus W}(b, c)$$

10 Oct 1

10.1 More Orientation

Remark. *Orientations of any two V, W $V \oplus W$ determine one on the third $V \oplus W \to W \oplus V, (\vec{v}, \vec{w}) \to (\vec{w}, \vec{v})$ when is this orientation preserving? Note that this is the determinant of a matrix that is the determinant of $\begin{pmatrix} 0 & I_{\dim \vec{v}} \\ I_{\dim \vec{w}} & 0 \end{pmatrix} = (-1)^{\dim V \dim W}$.*

Def. *If M is an oriented manifold-w-boundary then ∂M is oriented by the following recipe. $\mathcal{O}_{\partial M}(\vec{b}_1, \dots, b_{\dim M-1})$ where $\vec{b}$ is a basis of $T_m(\partial M)$. Note that we must extend it by one dimension, so we extend it orthogonally to the tangent space with another vector of norm 1. The book defines it as $\mathcal{O}_M(\textit{outward basis}, \vec{b}_1 \dots, \vec{b}_{\dim M-1})$.*

Remark. *Recall that $\partial^2 = 0$. We should look at the ker/image.*

Def. *$C_i = \{$formal linear combinations of diffeomorphism classes of compact i-mwb and oriented$\}$. We see that $C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \dots$. We see that $C_0 = \{m[+] + n[-]\}$, C_1 is the sum of n times the arrow from between two points ($\cdot \to \cdot$) and m times the circular arrow.*

Remark. *∂ of our $\cdot \to \cdot$ is $\cdot^- \ \cdot^+$. We see that $H_0 = \mathbb{Z}$ where $H_0 = \ker / \operatorname{im}$ at C_i. $H_1 = 0, H_2 = 0, H_3 = 0$ since each manifold is not the boundary of any manifold (???). $H_4 = \mathbb{Z}$ in particular has an example of the $\mathbb{CP}^2$ or the complex projective plane and is the ∂N^5. $H_5 = \mathbb{Z}/2$ since there is a manifold that is not the boundary but two of them is the boundary.*

10.2 Return to Intersection

Remark. *Recall that $I_2(f : X \to Y, Z \subseteq Y)$ X compact, $\dim X = \operatorname{codim} Z$ and this is the number of points in $f^{-1}(Z)$ after jiggling f to be $\pitchfork Z$ mod 2.*

Def. *$I(f$: compact $X \to Y, Z \subseteq Y)$ oriented manifolds with $\dim X = \operatorname{codim} Z$. This is defined as $\sum_{p \in f^{-1}(Z)} \operatorname{sign}(Tf(T_pX) \oplus T_{f(p)}Z$ versus $T_{f(p)}Y$.*

Ex. *If we have curve $(x+2)(x-2)x$ with standard orientation then at -2 it is negatively oriented (since the x vector is following the curve and the y vector is orthogonal), at 0 it is positive, and at 2 it is negative again.*

Def. *A **coorientation** of $Z \subseteq Y$ is an orientation on the vector spaces in $N_ZY = \{(z, \vec{v} \in T_zY) : \vec{v} \perp T_zZ\}$. We always have that $(N_ZY) \mid_z \oplus T_zZ = T_zY$. To visualize, note that in a sin curve the orientation vectors are perpendicular, since we want to know which side is $+$ which is $-$. The coorientation is orthogonal to the curve, and gives us which side "comes off" the manifold.*

Prop 10.1. *If $f : X \to Y$ and this factors through f_W ie $X \hookrightarrow W \xrightarrow{f_W}$ where $X = \partial W$ and is compact oriented and $Z \hookrightarrow Y$ is closed cooriented. Then $I(f, Z) = 0$.*

Proof. Jiggle f_W to be $\pitchfork Z$ also not changing $f_W \mid_X$ from f. Note that $f_W^{-1}(z)$ is an oriented 1 mwb since W is a mwb and $\partial f_W^{-1}(z) = f_W^{-1}(z) \cap \partial W$. Note that it is 1-mwb since we $\dim W = \operatorname{codim} Z$ and X is just one dimension less. We can also simplify $\partial W = X$ and so $\partial f_W^{-1}(z) = f^{-1}(z)$ is an oriented 0 manifold. Recall that 1-manifolds are the arrow or circular arrow and 0 manifolds are some number of positive and negative points. We have that $\sum \operatorname{sign}(pts) = 0$ and that is what we were computing. $\square$

Corrollary. *If $f_0, f_1 : X \to Y$ are homotopic and X is oriented compact and Z cooriented in Y. Then $I(f_0, Z) = I(f, Z)$.*

Proof. Apply the previous to $W = [0,1] \times X$ so $\partial w = -X \sqcup +X$ (where the signs indicated orientation). We see that $0 = I(\partial W, Z) - I(f_0, z) + I(f, z)$. $\square$

Def. *$f : X \to Y \hookleftarrow \{pt\} = Z$ where X is compact oriented and Y connected and oriented. Then $\deg f = I(f, pt)$.*

Theorem 10.1 (Fundamental Thm of Algebra)**.** *Any complex polynomial $p(z)$ of $\deg > 0$ has a root.*

Proof. Let $p_t(x) := \begin{cases} p(\frac{x}{t})t^{\deg p} & t \neq 0 \\ x^{\deg p} & t = 0 \end{cases}$. Assume p has no roots, then $p_{t\neq 0}$ has no roots. Let $\varphi_t : S^1 \to S^1 = \{z : |z| = 1\}$ which takes $z \to p_t(z)/|p_t(z)|$. We see that $\deg(\varphi_1) = \deg(\varphi_t > 0) = \deg(\varphi_0) = \deg p \neq 0$. So φ_1 doesn't extend to $D^2 \to \mathbb{C}$ (disc). But we had an extension $z \to \frac{p(z)}{|p(z)|}$, contradiction. $\square$

11 Oct 3

11.1 Review from last time

Remark. *We are working with $f : X \to Y \hookleftarrow Z$ where $\dim X = \operatorname{codim} Z$, X, Z compact X oriented Z cooriented (Y, Z both oriented).*

Def. *If $f \pitchfork Z, I(f, Z) := \sum_{x \in X, f(x) \in Z} \operatorname{sign}(Tf(T_xX) \oplus T_f(x)Z \xrightarrow{\sim} T_{f(x)}Y)$*

Theorem 11.1. *If $f \not\pitchfork Z$ can jiggle it to f' and $I(f', Z)$ doesn't depend on f'.*

11.2 Euler Characteristic based off of Orientation

Def. *If X is compact oriented, then $I(\triangle, \triangle) = \chi(X)$ is the Euler characteristic. Note that $\triangle(x) = (x, x)$.*

Def. *Paint(M) $:= \{(m \in M, o \in$ orientations of $T_mM)\}$. We can project using $\pi_1 :$ Paint(M) $\to M$ and this is a 2:1 map. The idea behind the paint is that if we paint the manifold and peel it off, for the sphere we have two spheres and for the Mobius strip we have one big strip of twice the size. Note that this paint is an abstract manifold, since we can double the charts on M and glue them together. So this paint comes with a canonical orientation.*

We define $\chi(M) := \chi($Paint(M)$)/2$.

Ex. $\chi(\mathbb{RP}^2) = 1$ *since the paint of $\mathbb{RP}^2$ is S^2. This is because for each pair of opposite points in the sphere is mapped to a line in $\mathbb{RP}^2$. $\chi(S^2) = 2$ (use the one from Morse theory but have not yet established $I(\triangle, triangle) = \chi$).*

11.3 Lefschetz Number

Remark. *One way to get another copy of X insides $X \times X$ which is hopefully $\pitchfork \triangle$ is as a graph of f which is $\{(a, f(a))\}$*

Def. *The **Lefschetz number** of $f : X \to X$ compact manifolds is $L(f) := I(graph f, \triangle)$.*

Ex. $L(\mathrm{id}_X) = \chi(X)$.

Def. *$f : X \to X$ is **Lefschetz** if $\mathrm{graph}(f) \pitchfork \triangle$ inside $X \times X$.*

Remark. *When is this true?. If for all $(a, b) \in \mathrm{graph}(f) \cap \triangle$ the tangent spaces add up. We need to consider the $a : f(a) = a$. So we have that $T_{(a,a)} \mathrm{graph}(f) + \triangle T_a X = T_{(a,a)}(X \times X)$ or equivalently that they intersect at 0 (by dimension). Note that $T_{(a,f(a))} \mathrm{graph}(f) = \mathrm{im}\, T_a g$ which is $g : a \to (a, f(a))$. Note that $T_a g : T_a X \to T_{(a,a)} X \times X$ is given by $\vec{v} \to (\vec{v}, T_a f(\vec{v}))$. In particular we want that $\{(\vec{v}, Tf(\vec{v}))\} \cap \{\vec{v}, \vec{v}\} = 0$. f is Lefschetz if at fixed point a of f $T_a f$ doesn't have 1 as an e-value. So in particular $T_a f - I$ is invertible.*

Note that $L(f) = \sum_{a:f(a)=a} \mathrm{sign}\det(T_a f - I)$

Ex. *$X = \mathbb{RP}^1 = \mathbb{R} \cup \infty = \{[x, y] \neq [0, 0]\}$ up to scale. $f : X \to X$ by $f(z) = 1.01z$ is homotopic to the identity on $\mathbb{RP}^1$. To compute the Lefschetz number note that the only fixed point 0 and that at that point $T_0 f - I = \big(0.01\big)$ which has determinant of sign $+$. So the Lefschetz number is 1.*

Ex. *$S^1 = \{e^{i\theta} \in \mathbb{C}\}$. Take $f_n(z) = z^n$, $n \in \mathbb{Z} \setminus \{1\}$. The $L(f_n) = \sum_{\mu_{n-1}} \mathrm{sign}\det(\big(n\big) - I) = n - 1$ since for the positive case it is $n - 1$ roots of unity (each contributing $+1$) and for $n < 1$ each term contributes -1 and there are $1 - n$.*

Corrollary. *No f_i is homotopic to f_j in the above case. This is because Lefschetz number if invariant under homotopy.*

Remark. *If $A, B \hookrightarrow M$ with $\dim A = \mathrm{codim}_M B$. If all oriented then $I(A, B) = (-1)^{\dim A \dim B} I(B, A)$. if $B = A$ and $\dim A \equiv 1 \pmod 2$ then $I(A, A) = 0$.*

Ex. *On the Mobius strip $I_2(A, A) = 1$ since M isn't orientable.*

Theorem 11.2. *If M compact oriented odd-dimensional then $\chi(M) = I(\triangle, \triangle) = -I(\triangle, \triangle) = 0$.*

12 Oct 8

12.1 Vector Bundles

Def. *A* ***vector bundle*** *over $M \subseteq \mathbb{R}^n$ is a submanifold B of $M \times \mathbb{R}^N$ s.t. $B \cap (m \times \mathbb{R}^N) \leq \mathbb{R}^N$ (linear subspace) for allm $\in M$ and is also of same dimension for each m.*

Ex. *$TM = \{(m, dx) : dx \in T_m M\}$. $M \hookrightarrow P$ then $N_M P$ is the normal bundle.*

Def. *A* ***section*** *of $B \to M$ is a map $M \to B$ s.t. $M \to B \to M$ is identity. In particular we map $m \to (m, \cdot)$.*

Ex. *$M \times \mathbb{R}$ is a vector bundle over M. A section over this is isomorphic a function $f : M \to \mathbb{R}$ with $m \to (m, f(m))$.*

Def. *If $B_1 \subseteq M \times \mathbb{R}^{N_1}, B_2 \times \mathbb{R}^{N_2}$ are vector bundles then $B_1 \oplus B_2 = \{(m, v_1, v_2) \in M \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}\}$ where $(m, v_i) \in M \times \mathbb{R}^{N_i}$.*

Def. *A map of vector bundles $B_1 \to B_2$ is a smooth map s.t.*

$$\begin{tikzcd} B_1 \arrow[r, "f"] \arrow[d] & B_2 \arrow[dl] \\ M & \end{tikzcd}$$

commutes and is linear for each $m \in M$. (ie it maps $m \to m$ and each vector linearly)

Theorem 12.1 (Tubular nbhd thm)**.** *If $Z \hookrightarrow Y$ and is closed, then $\exists$ an open $U \supseteq Z$ open in Y and a map $N_z Y \xrightarrow{\sim} U$ is local diffeomorphism around z*

Ex. *For Y a torus and Z a circle below the main one in the donut hole, the normal bundle is*

Proof. $N_z Y \to \mathbb{R}^N \xrightarrow{\text{closest point}} Y$ works only locally (near Y) $\square$

Remark. *Recall that $\chi(M) := I(M_\triangle, M_\triangle)$ both inside $M \times M$. Of course we need to jiggle $M_\triangle$ a little to make $M' \pitchfork M_\triangle$ to compute I. By the tubular nbhd theorem, we note that this is instead equal to $I(Z, Z)$ inside $N_{M_\triangle}(M \times M)$ where Z is the zero section. Note that the zero section is $M \to M \times \{0\}$.*

Theorem 12.2. *$N_{M_\triangle}(M \times M) \cong TM$ as bundles over M.*

Proof. Note that $(N_{M_\triangle}(M \times M))_m \hookrightarrow T_{(m,m)}(M \times M) \cong (T_mM)^{\oplus 2}$. Note that the normal bundle is perpindicular to $T_{m\times m}M_\triangle$ which is congruent T_mM which is embedded in $(T_mM)^{\oplus 2}$. Then we see that $V \overset{hookrightarrow}{\triangle} V \times V \hookleftarrow V^\perp$ and by projection to the first factor we see that $V^\perp \xrightarrow{\sim} V$ ☐

Corrollary (Poincaré-Hopf Thm)**.** *$\chi(M) = I(Z, Z)$ where Z is the zero section inside TM.*

Proof. Can perturb using a section of TM ie a vector field $\sigma : M \to TM$. So this is equal to the intersection number of the image of σ and the zero section $((m, \sigma(m))$ and $(m, 0))$ in TM and this is the equals to the sum over the zeros in σ of $(-1)^{\#\text{negative eigenvalues of } T_m\sigma}$

Near a 0 of σ,

$$\begin{tikzcd}
\sigma : M \arrow[r] & TM \\
\text{ball in } \mathbb{R}^n \arrow[u, hook] \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

So $T\sigma : T_mM \to T_{\vec{0}}(T_mM) \cong T_mM$. Let σ be generic enough that $T_m\sigma$ is $1 : 1$. ie the eigenvalues are non0. In particular we have that ${}_m\sigma$ is 1 and 2 block diagonalizable. Therefore the eigenvalues are $> 0, < 0$ or in pairs $z, \overline{z} \in \mathbb{C} \setminus \mathbb{R}$. ☐

Corrollary (Hairy Ball Thm)**.** *There doesn't exists a vector field σ on S^2 w no zeros because $\chi(S_2)$ is not 0, so we must have a zero of σ.*

Theorem 12.3. *If M has a nonvanishing vector field then $\chi(M) = 0$.*

Remark. *If M is connected, then the converse holds. IE $\chi(M) = 0$ implies that M has a nonvanishing vector field.*

Remark. *Recall that if $f : M \to \mathbb{R}$ was sometimes **Morse**.*

$$\begin{tikzcd}
M \arrow[r] \arrow[d, hook] & \mathbb{R} \\
\mathbb{R}^n \arrow[ur, "f"] &
\end{tikzcd}$$

Note that if ∇f is a vector field on $\mathbb{R}^n$, in particular the gradient. $\Sigma : M \to TM$ can be defined by $m \to (\nabla f)\mid_m$.

Ex. *For $S^2 \hookrightarrow \mathbb{R}^2 \xrightarrow{y} \mathbb{R}$. The gradient would be $+1$ everywhere. But σ takes the $+1$ on the tangent plane. To visualize at $(\pm 1, 0)$ it is still 1. At $(0, \pm 1)$ it is 0. Everywhere else is positive in the y direction. We see that if $f_1 = f_2$ on M then $\sigma_1 = \sigma_2$.*

Theorem 12.4. *This σ is nondegenerate in the Poincare-Hopf sense (isolated and $T\sigma$ is $1:1$ at the zeros) if f is Morse. In particular the zeros of σ are the critical points of f.*

Remark. *The set of 1 dimensional vector bundles with nonvanishing section over M is isomorphic to the set of 1-dim bundles $\cong$ to trivial line bundle $M \times \mathbb{R}$.*

Ex. *The radius bundle over S^2 is $N_{S^2}\mathbb{R}^3$. Note that $(T\mathbb{R}^3)\mid_{S^2} \cong S^2 \times \mathbb{R}$ and $N_{S^2}\mathbb{R}^3 \cong S^2 \times \mathbb{R}^3$. Note that TS^2 is neither trivial nor a line bundle.*

Remark. *When is TM trivial ($\cong M \times \mathbb{R}^{\dim M}$). For $M = S^k$. S^0, S^1, S^3, S^7 are the only ones. In these cases we identify each to S^{i+1} to the reals, complex, quaternions, and octonions. Then we imagine that each vector bundle is identified by some multiplication in the number system.*

13 Oct 10

13.1 Lefchetz Number Continued

Remark. *Recall that $L(f) = I(M_{\triangle}\,\mathrm{graph}(f))$ inside $M \times M$. Of course, this could be infinite, so we perturb f a little bit and get another graph and this is our Lefschetz number.*

Remark. *Claim: Small perturbation of $\mathrm{graph}(f)$ to another submanifold of $M \times M$ are of the form $\mathrm{graph}(g)$. We simply perturb f a little to g. We figured out when $M_{\triangle} \pitchfork \mathrm{graph}(f)$. In particular f is **Lefschetz** and isolated fixed points $\{p\}$ and $T_pf - I : T_pM \to T_pM$ is invertible.*

Remark. *If we only had that isolated fixed points, how do we get a local Lefschetz number?*

Ex. *$f : \mathbb{CP}^1 \to \mathbb{CP}^1 = \mathbb{C} \cup \infty$ by $f : z \to z^n$ where $n \neq 1$. The fixed points are 0 and ∞. Near 0, if we wiggle f then $z \to z^n$ has n solutions instead of 1 solution (0) of order n. These n solutions have Lefschetz number 1, so the total should be n. But we haven't figured this out yet!*

Remark. *Near p inside $B_\epsilon(p)$ in local coordinates, $f(z) = z$ only at p. For $z \in \partial B_\epsilon(p)$ we see that $\frac{f(z)-z}{|f(z)-z|} \in S^{k-1}$.*

Def. ***Local Lefschetz number*** *of f at p is the degree of the above map from $\partial B_\epsilon(p) \to S^{k-1}$.*

Remark. *Why is this homotopy invariant (the sum of p of the Local Lefschetz numbers) and why is this correct when f is Lefschetz? Well this is correct when f is Lefschetz by a linear algebra calculation. In particular $f - I$ shows up in the above map as the numberator and denominator. We get that* $\deg = \operatorname{sign}\det$.

For homotopy invariant, say f deforms to $\tilde{f}$ Leftchetz p to p_1, p_2. Let $p_1, p_2 \subseteq B_\epsilon(p)$. Since $\deg$ *is homotopy invariant we see that the degree. Note that* $\deg$ *is the only homotopy invariant map (Hopf-index theorem).*

13.2 Computing Euler Numbers

Def. *A* ***triangulation of*** *M is a decomposition $M = \bigsqcup_{i=0}^{\dim M} M_i$. M_i is the union of open i-simplices s.t. for each connected component C of $M_{\dim M}$, "facet", $\exists U \subseteq M$ open $\overline{C} \subseteq U$ s.t. we have standard simplex X and $X \subseteq U' \subseteq \mathbb{R}^m$ and U is diffeomorphic to u' and $\overline{C}$ is diffeomorphic to X. The standard simplex is $\{(x_1, \ldots, x_m) : x_i \geq 0, \sum x_i \leq 1\}$.*

Theorem 13.1. *$\chi(M) = \sum_i (-1)^i \#$face of dim i*

Theorem 13.2. *$\exists$ a vector field on k-simplex pointing toward center of i-simplices when i is even and away from the center of i-simplicies when i is odd.*

Remark. *Triangulations exist on smooth manifolds but not necessarily on topological manifolds. Note that on topological manifolds, we replace diffeomorphism with homeomorphism.*

13.3 Integration

Remark. *We want to show that for some smooth $f : M \to N$ we have $Tf : TM \to TN$ and want to find the integration $\int_M f$. What does this mean? When should one integrate it? Let's begin with smooth and compact support. Perhaps $U \underset{\varphi}{\xrightarrow{\sim}} M \xrightarrow{f} \mathbb{R}$. and $\int_M f := \int_U f \circ \varphi$. But this is no good since the reparametrization, this is bad. Think how $\int_0^{10} x 1 dx = 10$ but if $y = 2x$ we must reparametrize dx.*

Maybe integrate vector fields? Well, this is also not as clear. In the above idea, our vector field is 1 for x and we have to change it to $2x$ but this becomes hard.

Remark. *Recall: If* V *is a real vector space then* $V^\star = \{f : V \to \mathbb{R}\}$ *homomorphisms. Note that* $V \to (V^\star)^\star$ *by* $v \to f(v)$ *for* $f \in V^\star$*. Of course we could also compose this with a linear map. If* V *is finite dimensional, this map is invertible.*

Def. *A* $(0,1)$***-tensor*** *on* V *is an element of* $V^\star$*. A* 1***-form*** *on* M *is a section of* $T^\star M = \bigcup_{m\in M}\{m\} \times (T_m M)^\star$ *(cotangent bundle) and this is easily given structure of an abstract manifold, but not as easy to embed like the tangent bundle. Note that functions are sections of the trivial bundle by* $M \times \mathbb{R} \to M$*. A one form eats vector fields and gives a function.*

14 October 17

14.1 $(0,p)$-tensors

Def. *A* $(0,p)$*-tensors on* V *is a multilinear* $\alpha : V \times V \cdots \times V \to \mathbb{R}$ *where there are p V's. We are building it this way since we are trying to avoid the direct definition, but it is good enough for finite dimensional vector spaces. We won't need* (p,q) *tensors.*

Ex. $\mathbb{R}^n \times \mathbb{R}^n \to$ *matrices* $\xrightarrow{\sum m_{ij}} \mathbb{R}$*.* $(\vec{v}, \vec{w}) \to \vec{v}\vec{w}^T \to \sum_{c_{ij}}(v_i, w_j)$*. Each choice* $C = (c_{ij})$ *gives a* $(0,2)$ *tensor on* $\mathbb{R}^n$*, and these are all the* $(0,2)$ *tensors.*

Ex. $\det : (\mathbb{R}^n)^n \to \mathbb{R}$*. This is alternating (antisymmetric) ie it vanishes if two inputs are equal.*

Def. *If* α *is a* $(0,p)$*-tensor in* V*,*

$$\mathit{Alt}(\alpha)(\vec{v}_1, \ldots, \vec{v}_p) = \frac{1}{p!}\sum_{\sigma\in S_p}(-1)^{\ell(\sigma)}\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})$$

Where ℓ *is the number of inversions. Note that this definition is not standard, in fact we mostly don't include* $\frac{1}{p!}$ *since over a general field we can get division by* 0*.*

Def. *If* α *is a* $(0,p)$ *-tensor and* β *is a* $(0,q)$*-tensor then* $\alpha \times \beta$ *is a* $(0,p+q)$*-tensor by* $(\alpha \otimes \beta)(v_1, \ldots, v_p, w_1, \ldots, w_q) = \alpha(v)\beta(q)$*.*

Def. $\alpha \wedge \beta := \mathit{Alt}(\alpha \otimes \beta)$*.*

Theorem 14.1. *The space of alternating* $(0,p)$*-tensors of* $\mathbb{R}^n$ *has the basis* $\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} : 1 \le i_1 < \cdots < i_p \le n\}$ *(note there are* $\binom{n}{p}$ *choices) where* $dx_i : \mathbb{R}^n \to R$ *is projection to the i-th coordinate.*

Proof. Let α be an alternating $(0,p)$-tensor. We need to show $\exists!$ expansion $\alpha = \sum_{I \in \binom{[n]}{p}} c_I * (dx_{i_1} \wedge \cdots \wedge dx_{i_p})$.

Uniques We plug in e_{j_i} vectors to α and get the massive sum

$$\sum_I c_I \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^{\ell(\sigma)} dx_{i_{\sigma(1)}} e_{j_1} \dots dx_{i_{\sigma(p)}} e_{j_p}$$
$$= \sum_I c_I \frac{1}{p!} \sum_{\sigma : i_{\sigma(k)} = j_k} (-1)^{\ell(\sigma)} = \frac{c_J}{p!}$$

Therefore $c_J = p!\alpha(e_{j_1}, \dots, e_{j_p})$. To check that α is the same for any list of basis vectors, use multilinearity to reduce p-tuples of basis vectors then use alternating to reduce to $j_1 < j_2 \dots < j_p$. Therefore, c_J is unique. $\square$

Remark. *Note that $\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha$.*

14.2 Multivariable Change of Variable

Ex. *$(0, R) \times (0, 2\pi) \xrightarrow{\sim} D_R^0 \setminus (x \geq 0, 0)$ by $(r, \theta) \to r(\cos\theta, \sin\theta)$. For $U \xrightarrow{\sim} V \xrightarrow{a} \mathbb{R}$ we have that*

$$\int_V a dx_1 \dots dx_k = \int_U (a \circ f) |\det Tf| dx_1 \dots dx_K$$

This kind of sucks since we need to take the absolute value over the determinant of the Jacobian. For manifolds, we use forms to orient so we don't have to deal with this.

Def. *A p-**form** on a manifold M (not necessarily of dim p) is a smoothly varying choice of $(0,p)$-tensor on each $T_m M$.*

Ex. *$p = 0$ we have functions on M. $p = 1$ 1-forms are vector eater fields.*

For $f : M \to N$ we can define $f^\star : 0 - \text{forms on } N \to 0 - \text{forms on } M$ by $a \to a \circ f$ is the pushforward.

If α is a p-form on N then $(f^\star \alpha)(\vec{v}_1, \dots, \vec{v}_p) := \alpha(Tf(\vec{v}_1), \dots, Tf(\vec{v}_p))$. Furthermore $f^\star(\alpha \wedge \beta) = f^\star(\alpha) \wedge f^\star(\beta)$ where we wedge forms defined pointwise.

15 Oct 22

15.1 Integration on Manifolds

Theorem 15.1. *$f : U \to V$ where U, V open in $\mathbb{R}^k$. Then $\int_V g dx_1 \dots dx_k = \int_U g \circ f |\det(df)| dy_1 \dots dy_k$. Note that if f is orientation preserving then we don't need the absolute value. In particular $\int_V \omega = \int_U f^\star \omega$, where $\omega = g dx_1 dx_2 \dots dx_k$ and $f^\star \omega = g \circ f df_1 \dots df_k$, where $df_i = \sum_{j=1}^k \frac{\partial f_i}{\partial y_j} dy_j$ and so the above value is $g \circ f \wedge_{i=1}^k (\sum_{j=1}^k \frac{\partial f_i}{\partial y_j} dy_j) = g \circ f(\det(df)) dy_1 \wedge \dots \wedge dy_k$.*

Def. *X is a k dimensional manifold and ω a k-form. We say that ω is **compactly-supported** if $\{x \in X : \omega(x) \neq 0\}$ is compact in X.*

Def. *We define $\int_X \omega = \int_V \omega = \int_U \varphi^\star \omega$, where the support of ω is contained in an open set V and $\varphi : U \to \varphi$ is orientation preserving chart. We can see that it is well defined by using a change of variables from $\varphi_2^\star \omega$ to $\varphi_1^\star \omega$*

We could also define it using a partition of unity. If $\operatorname{supp}(\omega) \subseteq \bigcup_i U_i$. If we pick a partition of unity $\{\rho_i\}$ where each ρ_i has support contained in U_i, then we have $\int_X w = \sum_{j=1}^\infty \int_X \rho_i \omega$.

For submanifolds Z and ω is an ℓ-form and $\operatorname{supp}(\omega) \cap Z$ is compact in Z, , we can define $\int_Z \omega = \int_Z i^\star \omega$.

Ex. $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ *and* $\gamma : [0,1] \to \mathbb{R}^3, x \to (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ *and* $C = \gamma(i)$ *then*

$$\int_C \omega = \int_{[0,1]} \gamma^\star \omega = \int_0^1 \sum_{i=1}^3 f_i \circ \gamma_i \frac{d\gamma_i}{dx_i} dx_i$$

Ex. $F : [0,1] \times [0,1] \to R$ *where* $K = [0,1] \times [0,1]$ *and* $S = \{(x_1, x_2, F(x_1, x_2))\}$. *Then* $\omega = f_1 dx_1 \wedge dx_2 + f_2 dx_2 \wedge dx_3 + f_3 dx_3 \wedge dx_1$. *Then*

$$\int_S \omega = \int_K G^\star \omega$$

and we can calculate $G^\star \omega = f_1 \circ G dx_1 \wedge dx_2 + f_2 \circ G(-\frac{\partial F}{\partial x_1}) dx_1 dx_2 + (f_3 \circ G)(-\frac{\partial F}{\partial x_2}) dx_1 dx_2 = \left| f \circ G, \frac{\vec{n}}{|\vec{n}|} \right| |\vec{n}| dx_1 dx_2$, *where the form is dA or the area form.*

15.2 Exterior Derivative

Theorem 15.2. *There is a unique map d from p-forms to $p+1$ forms s.t.*

1. $d(w_1 + w_2) = dw_1 + dw_2$
2. $d(\omega \wedge \theta) = d\omega\theta + (-1)^p \omega \wedge d\theta$
3. $d(d\omega) = 0$.
4. $df = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} dx_i$ *when f is a 0-form.*
5. $g : X \to Y$ *smooth then* $g^\star \circ d = d \circ g^\star$.

In particular we define $\omega = \sum_i f_i dx_i \implies d\omega = \sum_i df_i \wedge dx_i$ *where ω is a k-forms and we sum over $I \subseteq [n]$ where $|I| = k$.*

Def. *We define the **curl** as* $\mathrm{curl}(f) = [-\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}]dx_1 \wedge dx_2 + [-\frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2}]dx_2 \wedge dx_3 + [-\frac{\partial f_1}{\partial x_3} + \frac{\partial f_3}{\partial x_1}]dx_1 \wedge dx_3$.

Def. *We can also define smooth map on manifolds.*

Ex. 0*-form* $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3$. 1*-form given by* $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ *has* $d\omega = \mathrm{curl}(\vec{F})$. *The 2-form* $\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$ *is given by* $d\omega = (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3})dx_1 dx_2 dx_3$ *is given by* $\mathrm{div}(f)$.

16 Oct 24

16.1 Stoke's Theorem

Theorem 16.1. *IF X is k-dim and ω is a $(k-1)$-form, then*

$$\int_X d\omega = \int_{\partial X} \omega$$

Proof. If $\mathrm{supp}(\omega) \cap \partial H^k = \emptyset$ then both sides are 0.

If $\mathrm{supp}(\omega) \cap \partial H^k \neq \emptyset$ where $\omega = \sum_{i=1}^{k} (-1)^{i-1} f_i dx_1 \wedge \dots \widehat{dx_i} \wedge \dots \wedge dx_k$ then

$$\begin{aligned}
\int_X d\omega &= \int_{H^k} \left(\sum_{i=1}^{k} \frac{\partial f_i}{\partial x_i} \right) dx_1 \dots dx_k \\
&= \int_{H^k} \frac{\partial f_k}{\partial x_k} dx_1 \dots dx_k \\
&= -\int_{\mathbb{R}^{k-1}} f(x_1, x_2, \dots, x_{k-1}, 0) dx_1 \dots dx_{k-1}
\end{aligned}$$

Define $\partial X = \{(x_1, \dots, x_{k-1}, 0) : x \in \mathbb{R}\}$. Given a unit normal to ∂X $-e_k = (0, \dots, 0, -1)$ then

$$\int_{\partial X} \omega = (-1)^K \int_{\partial X} \sum_{i=1}^{k} (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

For $i \neq k$, we note that

$$\int_{\mathbb{R}^k} (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k = 0$$

Since the k-coordinate is always 0. So we see that

$$\begin{aligned}
\int_{\partial X} \omega &= (-1)^k \int_{\partial X} (-1)^{k-1} f_k dx_1 \dots dx_k \\
&= \int_{\partial X} f_K dx_1 \dots dx_{k-1} = \int x d\omega
\end{aligned}$$

and we can use a partition of unity in the general case. □

Ex (Fundamental Theorem of Calculus). *f is a 0-form on $X = [a, b]$ and $\partial X = \{a, b\}$. Then $\int_X df = \int_{\partial X} f = f(b) - f(a)$.*

Ex (Green's Theorem). *$X = S \subseteq \mathbb{R}^2$ and $\partial X = C$. If $\omega = f_1 dx_1 + f_2 dx_2$ and so we ahve that*

$$\int_S d\omega = \int_C \omega \implies \int_S \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 dx_2 = \int_C f_1 dx_1 + f_2 dx_2$$

Ex. *$X \subseteq \mathbb{R}^3$ and $\omega = f_3 dx - 1 \wedge dx_2 + f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1$ and say $\partial X = S$. Then $d\omega = \mathrm{curl}(|F) \cdot \vec{u} dx_1 dx_2 dx_3$ then*

$$\int_X \mathrm{curl}(|F) \cdot \vec{u} dx_1 dx_2 dx_3 = \int_S \omega$$

16.2 De Rham Cohomology

Def. *X smoooth manifold then we have exterior derivative from k-forms to $k+1$-forms. A k-form ω is **closed** if $d\omega = 0$ and is **exact** if $\omega = d\theta$ for some $k-1$ form θ. The **k-thm dim De Rham cohomology group** $H^k(X) = \{$closed k forms$\}/\{$exact k-forms$\}$. It has a ring structure ???. Since $\omega_1 + d\theta_1 + \omega_2 + d\theta_2 = (\omega_1 + \omega_2) + (d\theta_1 + d\theta_2)$ and $\omega_1, \omega_2 \to \omega_1 \wedge \omega_2$*

Ex. *$\omega = \frac{-y}{x^2+y^2}dx = \frac{x}{x^2+y^2}dy$ is closed but not exact.*

Def. *$f : X \to Y$ given by $f^\# : H^k(Y) \to H^k(X)$ given by $f^\#\omega = f^\star\omega$ then $f^\star \circ d = d \circ f^\star$.*

Theorem 16.2. *$f : X \to Y$ then $\int_X f^\star\omega = \deg(f) \int_Y \omega$.*

Ex. *$f(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ given by $f : \mathbb{C} \to \mathbb{C}$. If $\arg(z) = \theta + 2\pi n$ where $z = re^{i\theta}$, then $d\arg$ is a well defined 1-form on $\mathbb{C} \setminus \{0\}$. we define $p = \frac{f}{|f|}$*

$$\int_{\mathbb{C}} p^\star d\arg = \deg(f) \int_{S^1} d\arg = 2\pi \deg(p)$$

which gives us the zeros inside our region Ω.

Proof. □

17 Oct 29

17.1 Operations on Forms

Remark. *We have $\wedge$ which takes $\alpha \wedge \beta$ and $d\alpha$ the exterior derivative. We also have $f^\star\alpha$ where $f : M \to N$. We note that, in fact we have some commutativity and distributivity guarantees, $f^\star d = df^\star$, $f^\star(\alpha\wedge\beta) = f^\star\alpha \wedge f^\star\beta$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg d \deg \alpha}\alpha \wedge d\beta$.*

17.2 Nullity and Rank Thm

Theorem 17.1 (Rank Nullity)**.** *If $T : V \to W$ is linear and V, W are finite dimensional vector spaces, then there are bases of v, w s.t. $T = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. Note that the rank is the number of 1's the the nullity is the number of 0's in the bottom left corner, and we recover that $\dim V$ is the height which is rank + nullity.*

Theorem 17.2 (Gabriel's Thm)**.** *For type A quivers ie let* $V_1 \xrightarrow{T_1} V_2 \ldots \xrightarrow{T_{d-1}} V_e$ *be a sequence of finite dimensional vector spaces and maps. Then there exists bases of all the* V_i *s.t. each* $T_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}$*. This means that basis elements are mapped into other basis elements until they go into the kernel.*

Remark. *What about complexes? Complexes are* $V_1 \xrightarrow{T_1} \ldots \xrightarrow{T_{e-1}} V_e$ *where* $T_{i+1} \circ T_i = 0$*. We see that* $H^i(\textit{complex}) = (\ker T_i)/(\operatorname{im} T_{i-1})$*, so* $H^\star$ *is the isolated bases vectors (not in the image and also in the kernel).*

Corrollary. $\sum_{i=1}^{e} (-1)^i \dim(V_i) = \sum_{i=1}^{e} (-1) \dim H^i$*. This is the Euler characteristic of complex/its cohomology.*

17.3 Goals for De Rham Cohomology

1. We wish to prove computability of De Rham Cohomology (Mayer-Vietoris).
2. Using that we wish to show that De Rham Cohomology of compact manifolds is finite dimensional.
3. If M is compact oriented we get a function $\int_M : H^\star(M) \twoheadrightarrow H^{\dim M}(M) \to \mathbb{R}$ which gives us a pairing $\langle \alpha, \beta \rangle \to \int_M \alpha \wedge \beta$. Our goal is to show that this pairing is **perfect** (ie nothing kills everything else).

Ex. *We use* $H^\star$ *practically.* $H^i(S^k) = \begin{cases} \mathbb{R} & i = 0, k \\ 0 & \textit{otherwise} \end{cases}$ *and* $H^i(B^{k+1}) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & \textit{otherwise} \end{cases}$*. Brouwer asked if there exists a map* $S^k \to B^{k+1} \to S^k$ *that is invertible. But if we apply the map to the cohomology then we are mappin*$H^k(S^k) \to H^k(B^{k+1})$ *that is invertible, which is from* $\mathbb{R} \to 0$*, so that is impossible.*

Remark. *Why is cohomology a ring?* $A \times B \twoheadrightarrow A, B$ *using* π_A, π_B*. THen* $H^\star(A) \xrightarrow{\pi_A^\star} H^\star(A \times B)$ *and similarly for* $\pi_B^\star$ *which means we have a map*$H^\star(A) \times H^\star(B) \to H^\star(A \times B)$*. If* $A = B$ *then* $H^\star(A \times A) \xrightarrow{\triangle^\star} H^\star(A)$ *given by* $A \hookrightarrow A \times A$ *for map* $\triangle_A$*.*

17.4 A Tool : Partition of Unity

Def. *A* ***partition of unity*** *on M is a countable or finite sum $\sum f_i = 1$ s.t. $f_i \geq 0$. This is a "locally finite sum" ie at any $m \in M$ the number of i s.t. $f_i(m) \neq 0$ is $< \infty$. Furthermore $\overline{\operatorname{supp}(f_i)}$ is compact.*

Def. *A partition of unity* ***subordinate to an open cover*** *$M = \bigcup U_i$ s.t. $\operatorname{supp}(f_i) \subseteq U_i$ ie same indexing set or equivalently we can find i s.t. $\operatorname{supp}(f_j) \subseteq U_i$ for all j.*

Theorem 17.3. *There exists a partition of unity subordinate to any open cover.*

Theorem 17.4 (Homotopy Invariance of $H^\star$)**.** *If $f_0 : f_1 : X \to Y$ are homotopic then $f_0^\star = f_1 * : H^\star(Y) \to H^\star(X)$.*

Proof. On forms, we're saying $\alpha \in \omega^p(Y)$ want $f_1^\star\alpha = f_0^\star\alpha = d\beta, \beta \in \Omega^{p-1}(x)$. Let $\beta(v_1, \ldots, v_{p-1}) = \int_{t=0}^1 F^\star\alpha(\frac{d}{dt}, v_1 \ldots v_{p-1})dt$. We will finish this next time. $\square$

18 Oct 31

18.1 Mayer-Vietoris

Def. *A* ***short exact sequence (SES)*** *of groups is $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ where $\ker(\beta) = \operatorname{im}(\alpha)$ and α is injective, β surjective.*

Def. *A* ***complex of*** *v* ***space*** *is a sequence $0 \to A_1 \xrightarrow{d} A_2 \ldots \xrightarrow{d} A_n \to 0$ where $d^2 = 0$.*

Def. *A* ***morphism of complexes*** *$A \to B$ is a sequence of f_i that map each $A_i \to B_i$.*

Theorem 18.1 (Mayer-Vietoris)**.** *A SES of complexes associated to $M = U \cup V$ open sets is*

$$0 \to \Omega^i(M) \xrightarrow{\textit{pullback}} \Omega^i(U) \oplus \Omega^i(V) \xrightarrow{\textit{difference of pullbacks}} \Omega^i(U \cap V) \to 0$$

To show $\Omega(U) \oplus \Omega(V) \to \Omega(U \cap V)$ is onto, hits some α, pick a partition of unity subordinate to V, V where $1_M = f_u + f_v$ and $\operatorname{supp}(f_U) \subseteq U$ and similar for f_V. $(\alpha f_i, -\alpha f_v) \to \alpha = \alpha(f_u) + \alpha(f_v)$.

Remark. *What do we get in $H^\star$.*

Theorem 18.2 (Homological Algebra)**.** *A SES of complexes a long exact sequence (LES) on $H^\star$. This LES given by $\to H^iA \to H^iB \to H^iC \to H^{i+1} \to H^{i+1}B \to H^{i+1}C$.*

Def. *The* ***Mayer-Vietoris LES*** *is $0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to \dots$. Eventually we get to $H^{\dim M}(U \cap V) \to 0$.*

Ex. *$S^1 = U \cup V$ where U is $(\cos t, \sin t)$ from $t = \pi/4 \to 7\pi/4$ and V is from $5\pi/4 \to 3\pi/4$. The MV LES is $0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M) \to 0 \to 0 \to 0$. $H^0(M)$ is 0-dimensional forms on M (connected components) so this sequence is $0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to ? \to 0 \to 0 \to 0$.*

Ex. *$S^n = (S^n \setminus \text{north}) \cup (S^n \setminus \text{south})$. Note that their intersection, the fat hemisphere, is homeomorphic to S^{n-1}. If we examine the sequence $0 \to H^0(S^n) \to H^0(D^1) \oplus H^0(D^2) \to H^0(D_1 \cap D_2) \to H^1(S^n) \to \dots$. Note that $H^i(D^1) \oplus H^i(D^2) = 0$. This means that $H^{i-1}(D_1 \cap D_2) \xrightarrow{\sim} H^i(S^n)$ since the surrounding parts are 0 so it must be isomorphic. We can see this because all three parts are similar to S^{n-1} and we know the ranks of the Cohomology groups of S^{n-1}.*

Ex. *$\mathbb{RP}^2$ (klein bottle). Let U be an open set in the center of the rectangle, V is the rectangle minus a small open set in the middle. Their intersection is a ring. They are isomorphic to a point, $\mathbb{RP}^1$ and S^1 respectively.*

The sequence is $0 \xrightarrow{0} H^0(\mathbb{RP}^2) \xrightarrow{1} H^0(D_1) \oplus H^0(D_2) \xrightarrow{1} H^0(D_1 \cap D_2) \xrightarrow{0} H^1(\mathbb{RP}^2) \xrightarrow{a} H^1(D_1) \oplus H^1(D_2) \xrightarrow{1-a} H^1(D_1 \cap D_2) \xrightarrow{a} \dots$, where we use the fact that $H^1(\{p\}) = 0$ and examine the ranks (which are $0 \to 1 \to 1+1 \to 1 \to a \to 0+1 \to 1$). So we need to examine one of these maps to get the other values. In particualr, $D_1 \cap D_2 \hookrightarrow D_2$ is the boundary of the mobius strip to the mobius strip. This is $d\theta \in \Omega^1(S^n)$, $\frac{xdy-ydx}{x^2+y^2}$ in $\mathbb{R}^2 \setminus 0$. In fact as we map from $H^1(D_2) \to H^1(D_1 \cap D_2)$ is by $\theta \to 2\theta$, so the rank of this map is 1. We see that $H^1(\mathbb{RP}^1) = H2(\mathbb{RP}^2) = 0$.

Proof. take an $\alpha \in A^i$ and note that $A^i \xrightarrow{f} B_i$ by $\alpha \to f(\alpha)$. Note that if we take an $\omega \in A^{i-1}$ then $\alpha + d\omega$ then it's equal to $f(a) + d\omega'$ and we are modding out by $d\omega$ so we are good. This define $H^iA \to H^iB$.

Another thing is that we can eventually find a $\gamma \in C^i$ s.t. $\gamma \to \alpha$. Then we define a $\beta in B^i$. Honestly, this proof is very long and a lot of diagram chasing. The basic learning moments from this theorem is that it's mostly just diagram chasing. Find a youtube "it's my turn" snake lemma. ☐

19 Nov 5

19.1 Geodesics

Def. *A* ***good cover*** $\{U_i\}$ *of* M *is a cover,* $\bigcup_{i\in I} = M$ *s.t. for all finite subsets* J $\bigcap_{j\in J} U_j$ *is contractible (homotopic to a point) or empty.*

Remark. *If* M *is compact and has a good cover, it has a finite good subcover.*

Remark. *Note that the set of* $J \subseteq I$ *s.t.* $\bigcup_J u_j \neq \emptyset$ *is closed under shrinkage. So this defines a simplical complex.*

Def. *A* ***geodesic on*** M *is a map* $\varphi : (a,b) \to M \hookrightarrow \mathbb{R}^n$ *that is as straight as possible over stretches in* (a,b)*. This means that it is infimum and this is a min in this case.*

Def. *An open set* $U \subseteq M$ *is* ***geodesically convex*** *if* $\forall u_1, u_2 \in U$*, there is a unique geodesic in* M $[0,1] \to [u_1, u_2]$ *and that geodesic is in* U*.*

Remark.

1. *Geodesically convex subsets are contractible.*

2. *A finite intersection of geodesically convex sets is geodesically convex.*

Theorem 19.1. *Around any* $m \in M$ *submanifold of* $\mathbb{R}^n$*, there is a geodesically convex nbhd.*

Proof. The geodesic spray is defined as $T_mM \to M$ takes each $\vec{v}$ to a flow time 1 along geodesic through m with derivative $\vec{v}$. This maps an open subset of T_mM to an open subset of M and this is our nbhd.dfc $\square$

19.2 More on forms and integration

Remark. *If* M *compact oriented manifold,* $H^\star(M)$ *finite dimensional, we have* $\int_M : \omega^{\dim M}(m) \hookrightarrow H^{\dim M}(M) \to \mathbb{R}$*. This gives us* $\langle \cdot, \cdot \rangle$ *by* $\langle \alpha, \beta \rangle = \int_M \alpha \cup \beta$ *where* $\alpha \in H^p(M)$ *and* $\beta \in H^{\dim M - p}(M)$*. Our goal is to show that this is a perfect pairing.*

Remark. *Distributions on Forms are called* ***Currents****. The cohomology of the currents is the same as the cohomology of manifolds.*

Def. $\Omega_c^p(M)$ *is the compactly supported p-forms on* M*. Can define* $\int_M : \Omega_c^{d\,\mathrm{im}\,M}(M) \to \mathbb{R}$*. We also see that* $d : \Omega_c^p(M) \to \Omega_c^{p+1}(M)$*. We define* $H_c^\star(M)$ *as* ***compactly supported cohomology****.*

Ex. $H_c^\star(\mathbb{R})$. *We see that* $0 \to \Omega_c^0(\mathbb{R}) \xrightarrow{d} \Omega_c^1(\mathbb{R}) \to 0$. *The kernel of the first* d *is constant functions. Since they are compact it follows that the only kernel elements is* $\{0\}$ *so* $H_c^0(\mathbb{R}) = 0$. *We examine* $t \to \int_{-\infty}^t \alpha$ *for* $\alpha \in \Omega_c^1(\mathbb{R})$. *We see that if we have some bump function the* $t \to \int_{-\infty}^t \alpha$ *is not finitely supported since for* $t \to \infty$ *the value is constant. But with some work we can see that* $H_c^1(\mathbb{R}) \cong \mathbb{R}$ *since any* β *we just define* $\beta - \int_{-\infty}^{\infty} /c\alpha_c = df$.

Remark. *Our new goal is to show that* $H^p(M) \times H_c^{\dim M - p}(M) \to H_c^{\dim M}(M) \xrightarrow{\int_M} \mathbb{R}$ *is a perfect pairing for oriented* M.

Remark. *Compactly supported cohomology is not homotopic invariant.*

Remark. Ω : *manifolds* $\to$ *supercommutative graded algebras given by* $f : M \to N \implies f^\star : \Omega(N) \to \Omega(M)$. *What about* Ω_c*? It is covariant for open inclusions* $\Omega^p(u) \ni \alpha \to$ *extend by* 0 *off of* $U \in \Omega^p(M)$. *This only works because* α *is of compact support in* U.

19.3 Mayer-Vietoris for $H_c^\star$

Theorem 19.2. $M = U \cup V$ *open. Then the sequence is* $\Omega_c^\cdot(U \cap V) \to \Omega_c^\cdot(U) \oplus \Omega_c^\cdot(V) \xrightarrow{difference} \Omega_c^\cdot(M) \to 0$. *We can see this is exact.*

Ex. $M - V$ *for* $H_c^\star(S^1)$ *where* U *and* V *are large arcs.*

	$U \cap V$	$U \oplus V$	M
H_c^0	0	$0 \oplus 0$	?
H_c^1	$\mathbb{R}^2$	$\mathbb{R} \oplus \mathbb{R}$	?
H_c^2	0	0	0

we can find a map from $H_c^1(U \cap V) \to H_c^1(U) \oplus H_c^1(V)$ *that is of rank* 1. *Therefore, we have that both ? are of dimension* 1 *and we can see that they are the same.*

Remark. *Still need base case* $H_c^\star(\mathbb{R}^n)$. *Covariance* $M \times N \xrightarrow{\int_N} M$ *by* $\Omega_c^p(M \times N) \to \Omega_c^{p - \dim N}(M)$.

Warmup $H^p(M \times \mathbb{R}) \cong H^p(M)$. *This is true because* $M \times \mathbb{R} \xrightarrow{\pi}$ *and* $m \xrightarrow{\eta} (m, 0)$ *is the reverse.* $\pi^\star \circ \eta^\star : \Omega^\cdot(M \times \mathbb{R}) \to \Omega^\cdot(M \times \mathbb{R})$ *want* $1 - \pi^\star \circ \eta^\star$ *to be* d *on closed forms. Need "homotopy operator" of degree* -1.

20 Nov 7

20.1 Integration and Homotopy

Remark. *Our goal is to show that if M is oriented and non compact then $H^i(M) \times H_c^{\dim M - i}(M) \to H_c^{\dim M}(M) \xrightarrow{\int} \mathbb{R}$ is a perfect pairing.*

A warmup is given an operator $\eta : M \to M \times \mathbb{R}$ with inverse $\pi : M\ times\mathbb{R} \to M$

$$\begin{tikzcd}
\Omega^i(M) \arrow[r] \arrow[d, leftrightarrow] & \Omega^{i+1}(M) \arrow[d] \\
\Omega^i(M \times \mathbb{R}) \arrow[r] & \Omega^{i+1}(M \times R)
\end{tikzcd}$$

The trick is that there exists a K "Homotopy Operator" s.t. $1 - \eta^\star \pi^\star = H^{q-1}(dK - Kd)$ on q-forms. We define $K(\alpha)(x,y) := \int_{s=0}^t \alpha(\frac{d}{dt}, \dots)$ where α is a q-form, $x \in M, t \in \mathbb{R}$.

Lemma. $H_c^{i+1}(M \times \mathbb{R}) \cong H_c^i(M)$.

Proof. We note that $\Omega_c^{i+1}(M \times \mathbb{R}) \xrightarrow{\int} \Omega_c^i(M)$ and with reverse $\alpha \to (\pi^\star \alpha) \wedge$ (fixed bump form on $\mathbb{R}$). We want a homotopy operator K s.t. $1 - (e \wedge \pi^\star) \circ \int_{\mathbb{R}} = (-1)^{q-1}(dK - Kd)$. We set $K(\alpha)(x,y) = \int_{-\infty}^t \alpha(\frac{d}{dt}, \dots) - \int_{-\infty}^t e \int_{-\infty}^{\infty} \alpha(\frac{d}{dt}, \dots)$. K satisfies the equation $1 - \pi^\star \circ \eta^\star = (-1)^{q-1}(dK - Kd)$ and so is a homotopy. $\square$

Corrollary (Poincaré Duality)**.** $H_c^i(\mathbb{R}^n) = \begin{cases} \mathbb{R} & i = n \\ 0 & i \neq n \end{cases}$. *Recall that the original statement was $H^i(M) \times H_c^{\dim M - i}(M) \xrightarrow{\int \circ \wedge} \mathbb{R}$ is perfect. Note that htis only occurs for M oriented with finite good cover.*

20.2 M-V in $H^\star$

Remark. *Suppose $H^k = U \cup V$ and recall the long M-V sequence $\cdots \to H^i(M) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \xrightarrow{\partial} H^{i+1}(M) \to \dots$. We note that by duality that $H_c^{k-i}(M)^\star \to H_c^{k-i}(U)^\star \oplus H_c^{k-i}(V)^\star \to H_c^{k-i}(U \cap V)^\star \to H_c^{k-i-1}(M)^\star$. We also assume that this manifold has finite good cover and is oriented. In particular we have isomorphisms*

$$\begin{array}{ccccccc} \to H^i(M) & \longrightarrow & H^i(U)\oplus H^i(V) & \longrightarrow & H^i(U\cap V) & \xrightarrow{\partial} & H^{i+1}(M) \\ \downarrow & & \downarrow\wr & & \downarrow & & \downarrow \\ H_c^{k-i}(M)^\star & \to & H_c^{k-1}(U)^\star H_c^{k-i}(V)^\star & \to & H_c^{k-i}(U\cap V)^\star & \to & H_c^{k-i-1}(M)^\star \end{array}$$

and to show that all are isomorphism we just apply the Five Lemma.

Lemma (The Five Lemma)**.** *If we have a exact sequence morphism*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Then $C \to C'$ is also an isomorphism.

Corrollary. *If M is compact oriented connected, then $H_c^{\dim M} \cong \mathbb{R}$.*

20.3 Kunneth

Def. *R is a commutative ring with M, N modules. Then $M \otimes_R N$ is the free module on (m, n) quotient by $(m_1 + m_2, n) = (m_1, n) + (m_2, n)$ and $(rm, n) = r(m, n) = (m, rn)$.*

Ex. $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \cong \mathbb{Z}/2$. *Notice that the free module is sums $\sum_{finite} r_{ab}(a, b)$ but since we quoteitenting out by stuff like* $6(1,1) = (6,1) = (0,1) = 0(1,1)$.

$\mathbb{R}_\otimes \mathbb{R} \cong \mathbb{R}$ *and in general $\mathbb{R}^a \otimes_R \mathbb{R}^b \cong \mathbb{R}^{ab}$*

Remark. *Bilinear $(M \times N, \cdot)$ set of bilinear maps. We see that R-Modules $\xrightarrow{Covariant}$ Set by $A \to$ Bilinear$(M \times N \to A)$ and if we compose both with φ then we map $B \to$ Bilinear$(M \times N \to B)$ which gives us the functor aspect. In general the basic functors are Hom$(T, \cdot)$ that maps $A \to$ Hom$_R(T, A)$ and these are representable functors. The Tensor product gives us this information.*

Theorem 20.1 (Kunneth)**.** *Let M, N be manifolds where M has a finite good cover. Then the natural map $H^\star(M) \times H^\star(N) \to H^\star(M \times N)$ by $\sum_i \alpha_I \otimes \beta_i \to \sum_i (\pi^\star \alpha_i) \vee (\pi^\star \alpha_i)$.*

Proof. Involved (Refer to book). □

21 Nov 12

21.1 Revisiting Vector Bundles

Def. *A* ***vector bundle*** *V on M is a map from $M \to V$ with scalar multiplication and addition and a zero section satisfying linear axioms.*

Def. *The* ***fiber product*** *$A \times_M B$ of $A \xrightarrow{f_A} M, B \xrightarrow{f_B} M$ is $\{(a, b) \in A \times B : f_A(a) = f_B(b)\}$.*

Theorem 21.1 (Tubular Neighborhood)**.** *If $m \hookrightarrow 0$ then there exists $U \supseteq M$, $U \subseteq 0$ open with $U \cong N_M 0$ the normal bundle.*

Def. *An* ***oriented vector bundle*** *has an orientation on each fiber continuously.*

Def. *A* ***Thom form*** *on an oriented vector bundle $V^{n+k} \xrightarrow{\pi} M^n$ is a closed form β on V s.t. for all $m \in M$ $\beta \mid_{\pi^{-1}(m)}$ is a compactly supported bump form with integral equal to 1.*

Ex. *Take $V = \mathbb{R}^2 \hookleftarrow \{(x_0, *)\} = \{x_0\} \times \mathbb{R}$ where $\beta = f(x, y)dx + g(x, y)dy$. Then we see that on $\{x_0\} \times \mathbb{R}$ then $f(x, y)dx$ vanishes so we only know that $g(x, y)dy$ is a bump form for all x.*

Theorem 21.2 (Thom)**.** *$H_c^\star(M) \cong H_c^{\star+k}(V)$ where V is an oriented k-plane bundle over M. This is dual to $H^\star(M) = H^\star(V)$ if M is compact oriented.*

Proof. The first part of this proof is that there exists a Thom form β on V constructed normally when $M = \mathbb{R}^n$ once and for all, and glued together using a partition of unity, subordinate to an open cover $\{U\}$ on which V is a trivial vector bundle. $V \mid_U \cong U \times \mathbb{R}^k$. Take the forms β_U and $\beta = \sum f_U \beta_U$.

For the second part, we take a map from $\Omega_c^{\cdot}(M) \to \Omega_c^{\cdot+k}(V)$ by $\alpha \to (\pi^\star\alpha) \wedge \beta$. We can construct the inverse by $\int_{fiber}$. Therefore there exists a homotopy operator that makes each forward and backwards map homotopic to the identity.

On V we can define a weird complex of forms φ each $\varphi/\pi^{-1}(m) \in \Omega_c(\pi^{-1}(m))$ of "compactly supported in vertical directions" and we have $\Omega^{\cdot}(M) \to \Omega^{\cdot+k}_{vertical}(v)$ by $\alpha \to \pi^\star(\alpha) \wedge \beta$. $[m]$ is the **Poincaré Dual of** $M \subseteq O$. $\square$

Ex.

1. *$M \hookrightarrow V$ where $m \to (m, \vec{0})$ an oriented vector bundle. We get the Thom form $\beta \in \Omega^k_{vertical}(V)$ where $\sigma^{-1}(\beta) \in \Omega^k(M)$ closed. Then $[\sigma^\star(\beta)] = \sigma^\star[\beta]$ is the Euler class of V and is independent of β.*

2. *$M \hookrightarrow O$. M os compact and both are oriented. We get that $\int_M : H^{\dim M}(O) \to \mathbb{R}$. We have the sequence $M \hookrightarrow N_M O \cong U \hookrightarrow O$ where U is the tubular nbhd of M. Thom form, extended by 0 represents $[m]$. We claim that*

$$\int_O \eta^\star(\beta \text{ Thom form on } N_M O) \wedge \alpha = \int_M \alpha|_M = \sum \int_W (\alpha + f_W)|_W$$

Note that we find this since this is equal to

$$\int_U \beta \wedge \alpha = \int_{N_M O} \beta \wedge \alpha = \sum_W \int_{\pi^{-1}(W) \cong W \times \mathbb{R}^k} \beta \wedge (\alpha f_w)$$

and we just apply fubini to get our desired since β becomes nothing. We use a partition of unity M subordinate to an atlas on which $N_M O$ is trivial, to replace α by $\alpha \cdot f_w$.

Theorem 21.3. *If $M_1, M_2 \hookrightarrow O$ all compact oriented and $M_1 \pitchfork M_2$ then $[M_1][M_2] = [M_1 \cap M_2]$. The product is cup product (wedge product on duals).*

Proof. Put a tubular nbhd around $M_1 \cap M_2$ that is equivalent to $N_{M_1 \cap M_2} O$ where $N_{M_1 \cap M_2} M_i \subseteq N_{M_1 \cap M_2} O$. The two $N_{M_1 \cap M_2} M_i$ are transverse in $N_{M_1 \cap M_2} O$. Now fiber by fiber we're looking for bump forms β_1 β_2 s.t. $\pi_1^\star(\beta_1) \wedge \pi_2^\star(\beta_2)$ is a bump form on $V_1 \times V_2$. □

22 Nov 14

22.1 Lefschetz Numbers and Cohomology

Remark. *Recall that given $f : M^k \to M$ compact oriented then the Lefschetz number if $L(f) = I(\text{graph}(f), \triangle) = \int_{M \times M} [\text{graph}(f)] \vee [M_\triangle]$.*

Theorem 22.1.

$$\begin{array}{ccc} H^a(M) \otimes H^b(M) & \xrightarrow{\text{kunneth}} & H^{a+b}(M \times M) \\ \Big\downarrow \text{not hom} & & \Big\downarrow \int_{M_\triangle} \\ \mathit{end}(H^\star(M)) & \xrightarrow{\quad\text{trace}\quad} & \mathbb{R} \end{array}$$

Def. *The* β***-transform*** $\Phi_\beta : H^\star(M) \to H^\star(N)$ *by* $\alpha \to \int_M (\pi_M^\star(\alpha) \vee \beta)$ *where* $\beta \in H^\star(M \times N)$ *and both* M *and* N *are compact oriented.*

Ex. *Recall the Fourier transform . We can write it as* $\widehat{f}(w) = \int_V f(\vec{v}) e^{i\langle \vec{v}, w\rangle}$ *and this is analogous to the* β*-transform. Note that as opposed to our normal requirement of analysis, since we make things compact then this is well defined always.*

Remark. *Our goal is that if* $f : M \to M$ *which is compact oriented then* $L(f) = \sum_{i=0}^{\dim M} (-1)^i tr(f^\star : H^i(M) \to H^i(M))$*. The corollary is that* $\chi(M) = \sum (-1)^i \dim H^i(M)$*.*

Lemma. *Given* $f : N \to M$ *we have that* $f^\star : H^\star(M) \to H^\star(N)$*. We claim that* $f^\star = \Phi_{[\mathrm{graph}\, f]^T \subseteq M \times N}$*.*

Proof.

$$\int_N \Phi_{[(\mathrm{graph}\, f)^T]}(\alpha)\beta \int_N \beta \int_M \pi_M^\star(\alpha) \vee [(\mathrm{graph}\, f)^T]$$

$$= \int_{M \times N} (\alpha \otimes \beta)[(\mathrm{graph}\, f)^T] = \int_{(\mathrm{graph}\, f)^T} \alpha \otimes \beta = \int_N f^\star(\alpha)\beta$$

□

Def. *The Lefschetz formula for* $\beta \in H^\star(M \times M)$ *is given by* $\int_{M \times M} \beta \vee [\triangle(M)] = \int_M \triangle^\star \beta$ *where* $\triangle : M \to M \times M$*. We define this as* $\sum_{i=0}^{\dim M} (-1)^i tr(\Phi_\beta : H^i(M) \to H^i(M))$*. Both sides are linear in* β*. Enough to check for* $\beta = \alpha_i \otimes \alpha^j$ *the basis and dual basis elements.*

Ex. *If we take* $\beta = \alpha_I \otimes \alpha^j$ *then we get that* $\int_{M \times M} \beta \vee [\triangle(M)] = \int_M \alpha_i \alpha^j = [i = j]$*. However, for the right hand side of our above equation, we have that this is* $\sum_{\alpha^\ell} (-1)^{\deg(\alpha^\ell)} \left\langle \Phi_{\alpha_i \otimes \alpha^j}(\alpha^\ell), \alpha_\ell \right\rangle = \int_M (\alpha^\ell \otimes \cdot (\alpha_o \otimes \alpha^j)) = [i = \ell]\alpha^j = [i = \ell][j = \ell] = (-1)^{\deg(\alpha_i)}[i = j]$*. We didn't find where the* -1 *comes from or how to remove it.*

Remark. *In our definition of* $L(f)$ *and the Euler characteristic, we note that the supertrace we define does not need that* M *is compact oriented. Does this mean we can define it for non compact manifolds? How much do the Betti Numbers* $\dim H^i$ *affect our actual understanding?*

23 Nov 19

23.1 More Lefschetz

Remark. *Recall that if* M *is compact oriented and* $f : M \to M$ *is Lefschetz (isolated fixed points enough) that we have*

$$\sum_{m:f(m)=m} \textit{local Lefschetz number} = \sum_i (-1)^i tr(f^* : H^i(M) \to H^i(M))$$

Ex. *We showcase an example on $\mathbb{CP}^n$. We can calculate to get that the cohomologies are of dimension 1. In particular we get that $\mathbb{R}[h^{(2)}]/\langle h^{n+1}\rangle = \mathbb{R} \oplus \mathbb{R}h \oplus \cdots \oplus \mathbb{R}h^n$.*

23.2 Lie Groups

Def. *A* ***Lie group*** *G is a group that is the manifold.*

Ex. *$GL(n, \mathbb{R})$ is an example of a Lie group. This is because the determinant of AB is the composition of the determinant of A and B. This is the* ***general linear group****.*

Remark. *If G is a closed subset of $GL(n, \mathbb{R})$, then G is a Lie group. If G is connected and $H \leq G$ is discrete and normal, it is* ***central*** *if it commutes with all of G ie $ghg^{-1} = h$ for all g and $h \in H$.*

Ex. *$SL(n, \mathbb{R})$ special linear group, $GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R})$ are good examples. So are $O(n, \mathbb{R})$ the orthogonal matrices and $SO(n, \mathbb{R})$ the special orthogonal ones. Another good example is $GL(n, \mathbb{H}) \subseteq GL(4n, \mathbb{R})$ and well as the unitary group $U_n = \{M \in GL(n, \mathbb{C}) : M\overline{M^T} = I\}$.*

Remark. *If G is a topological group that is compact and simple, then it is Lie.*

Theorem 23.1. *Let G_0 be a connected component containing $I \in G$ (an example of is $O(n, \mathbb{R})_0 = SO(n, \mathbb{R})$). Then G_0 is normal in G.*

Proof. If $h \in G_0$ there is a path $\gamma : [0, 1] \to G_0$ with $0 \to I$ and $1 \to h$. If we look at $t \to g^{-1}\gamma(t)g$ then $0 \to I$ and $1 \to g^{-1}hg \in G_0$ by our definitions. □

Def. *A* ***Lie group action*** *$G : M \to M$ where M is a manifold is a map $G \times M \xrightarrow{\alpha} M$ smooth ($\alpha M = M$) s.t.*

$$\begin{tikzcd}
G \times G \times M \arrow[r, "I_\alpha \times \alpha"] \arrow[d, "\cdot_G \times I_m"] & G \times M \arrow[d, "\alpha"] \\
G \times M \arrow[r, "\alpha"] & M
\end{tikzcd}$$

Remark. *We want that when G acts on M that M/G is again a manifold. In particular we need the action to be free and proper.*

Ex. *Good examples of cases where if we don't have free and proper that don't work are if $\mathbb{R}$ acts on $T^2 = \{(w,z) \in \mathbb{C} : |w| = |z| = 1\}$ by $s \cdot (w,z) = (e^{is}w, e^{is\sqrt{2}})$ is free but not proper. In particular it is a ray on the torus (in the square identification) and is dense but is not a manifold. This is because it is free but not proper.*

If $\mathbb{R}^\times$ acts on $\mathbb{R}^2 \setminus 0$ by $z \cdot (x,y) = (zx, z^{-1}y)$. $(\mathbb{R}^2 \setminus 0)/\mathbb{R}^\times$ is $\mathbb{R}$ with two origins, so this is not a manifold since it is not proper (it is free).

Def. *An action is **free** if for all $g \neq 1$ then $g \cdot m \neq m$ for all m. In particular this means that the stabilizer of each m is I_G. An action is **proper** if $(g,m) \to (m, g{\cdot}m)$ is proper (X compact implies preimage is also compact).*

Theorem 23.2. *If G acts on M properly and freely then M/G is a manifold.*

24 Nov 21

24.1 More Lie Groups

Remark. *Let G, H be Lie Groups (groups and manifolds). Examine $\varphi : G \to H$. Then we note that $T\varphi : T_gG \to T_{\varphi(g)}H$ is the tangent map. For the tangent space T_gG, we note that this is isomorphic to T_1G by map $g\cdot : G \to G$ (we use the tangent space map). As a corollary this parallelizes the tangent bundle to G and that S^{2n} is not a group by hairy ball theorem. Let $T_1G = \mathrm{Lie}(G)$.*

Remark. *If $G = \mathbb{R}$ and $\varphi : \mathbb{R} \to H$, then φ is determined by $T_1\varphi : \mathbb{R} \to \mathrm{Lie}(H)$ or by $T_1\varphi(1) \in \mathrm{Lie}(H)$. The H is the **one parameter subgroup** and $T_1\varphi(1)$ is the **infinitesimal generator**.*

*Given $x \in \mathrm{Lie}(H)$ there exists a unique $\varphi : \mathbb{R} \to H$ by $T_1\varphi(1) = x$. If G is connected and $\varphi : G \to H$ is a smooth homomorphism, then φ is uniquely determined by $T_1\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. We define the **lie exponential map** $\exp : \mathrm{Lie}(G) \to G$ by $x \to \varphi_x(1)$.*

We claim that any open $U \subseteq G$ with $1 \in U \neq \emptyset, U = U^{-1}$ generates G. To do so we note that $H = \bigcup_{i \in \mathbb{N}} U^i$ and so any open subgroup in G is also closed, where U^i are the cosets.

Remark. *If $\rho : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ respects the Lie bracket and G is connected and simply connected then there exists a unique $\varphi : G \to H$ given by $T_1\varphi = \rho$.*

Ex. *We note that if we have lie group homomorphism $S^1 \to \mathbb{R}$ then we note that since S^1 is compact it's image is compact. But the only compact subgroup of $\mathbb{R}$ is $\{0\}$. Similarly, when we map $S^1 \to S^1$ we see that the map must be of the form $z \to z^n$ for some integer n.*

Many of the examples come from quantum mechanics. In particular, we note that for $\varphi : SO(3) \to U(\mathcal{H})$ then does this map come from $so(3) \to u(\mathcal{H})$ the Lie Algebras. We note that $SO(3)$ is not simply connected so this only occurs half the time.

24.2 Some Representation Theory

Def. *If $\varphi_1, \varphi_2 : G \to GL(\mathbb{C}^n)$ then there exists $M \in GL(\mathbb{C})$ s.t. $\varphi_1(g) = M\varphi_2(G)M^{-1}$ iff for all $g \in G$ $Tr\varphi_1(g) = Tr\varphi_2(G)$ where this is the **character of** φ_1. Now we examine two seemingly contradictory theorems.*

Theorem 24.1. *If G is finite and $H \leq G$ and $H \neq G$ then H misses some conjugation class of G ie there exists some g s.t. for all k $kgk^{-1} \notin H$.*

Theorem 24.2. *Unitary matrices are diagonalizable.*

24.3 Quotient Manifold Theorem

Theorem 24.3 (Quotient Manifold Thm). *If G is a Lie group and G acts on M freely and properly ie for all $m, g \cdot m = m \implies g = 1$ and $(g, m) \to (m, gm)$ is proper, then*

1. *Each orbit is a closed submanifold*
2. *$M/G := M/\sim$ is a manifold s.t. $M \to M/G$ is a submersion.*

Def. *A **principal bundle** G when $E \xrightarrow{/G} M$ is a*

1. *a space E with free proper G-action.*
2. *diffeomorphism ie $E/G \cong M$. In particular each fiber is diffeomorphic to G as a G-space.*

Ex. $S^1 = \{e^{i\theta}\} \subseteq \mathbb{C}$. *$\varphi : S^1 \to S^1, z \to z^n$ then we note that $\mathbb{Z}/n$ acts on S^1 by $k \cdot z = (e^{\frac{2\pi i}{n}})^k z$. If we take $S^1 \to S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ then we can construct the action $e^{i\theta} \cdot (z, w) = (e^{i\theta} z, e^{i\theta} w)$ satisfies our conditions. In this case $G \to G \times M \to M$ is a trivial principal G-bundle.*

Ex. *What are the $\mathbb{Z}/2$-bundles on S^1? We see that it is either φ_1 or φ_2 the trivial and nontrivial bundles.*

Ex. *M^k then the Frame$(M) = \{(m, \vec{v}_1, \dots, \vec{v}_m) : (\vec{v}_i)$ basis of $T_mM\} \subseteq \prod_{i=1}^k TM$. We can act on this by $GL(\mathbb{R})$ by replacing our list of vectors, which makes the frame a principal $GL_k(\mathbb{R})$ bundle.*

Def. *Let $G \to E \to M$ be a principal G-bundle. Let G act on X smoothly. The **associated X-bundle** $(X \times E)/G_{diagonalization}$ is a nonprincipal bundle.*

Ex. *$\mathbb{Z}/2 \to S^1 = E \to S^1 = M$ via φ_2. Let $\mathbb{Z}_2$ act on $\mathbb{R}$ by I ie $k \cdot \vec{v} = (-1)^k \vec{v}$. What is the associated $\mathbb{R}$ bundle?. Well it is $\mathbb{R} \to (\mathbb{R} \times S^1)/\mathbb{Z}_2 \to S^1$. Simply replace the $\mathbb{Z}_2$ we had with reals and this forms the Mobius band.*

Ex. *M^k by $GL_k(\mathbb{R})$ acts on $\mathbb{R}^k$ then the associated bundle is given by $(\mathbb{R}^k \times$ Frame$(M))/GK_k(\mathbb{R})$, which is just the tangent bundle. What is the map?*

Ex. *Examine $GL_k(\mathbb{R})$ acting on $(Alt^p\mathbb{R}^k)^\star$ and sections of associated bundle are $\Omega^p(M)$.*

Ex. *$GL_k(\mathbb{R})$ acts on $\mathbb{R}$ by $M \to |\det M^{-1}|$ are called top forms are better versions of*

25 Nov 26

25.1 Culture

Def. *$K^0(M)$ is defined as the finite differences of isomorphism classes of $\mathbb{C}$-vector bundles on M.*

Theorem 25.1 (Chern Character)**.** *$\mathbb{R} \otimes_{\mathbb{Z}} K^0(M) \xrightarrow{\sim} H^\star(M)$.*

25.2 $H^\star(\mathbb{CP}^n)$

Remark. *Recall that $\mathbb{CP}^n = \{z \in \mathbb{C}^n, z \neq 0\}/\mathbb{C}^\times = \{z : |z| = 1\}/U(1)$ where $U(1) = \{[e^{i\theta}]\}$. We see that $\mathbb{CP}^n \xrightarrow{\sim}$ Hermitian matrices M with eigenvalues $= 1, 0, \dots, 0$ by $z \to z^* z$ where z^* is the conjugate transpose.*

We define $\mathbb{CP}^n = \{[z_0, \dots, z_n] : z_n = 1\} \cup \{[z_0, \dots, z_n], z_0, \dots, z_{n-1}$ not all $0\}$. We see that $U = \mathbb{C}^n$ and $V = \mathbb{CP}^{n-1}$ with intersection $\mathbb{C}^n \setminus \{0\} \cong S^{2n-1}$.

We construct the Mayer Vietoris sequence. Note that near the end it becomes

	$\mathbb{CP}^{n>1}$	$U \amalg V$	$U \cap V$
H^{2n-3}	?	0 0	0
H^{2n-2}	?	0 1	0
H^{2n-1}	?	0 0	1
H^{2n}	?	0 0	0
H^{2n+1}	0	0 0	0

We can fill up the above. To get that $H^*(\mathbb{CP}^n) = \mathbb{R}[h^{(2)}]/\langle h^{n+1}\rangle$

Taking $h \to$ *the Poincaré dual of hyperplane gives us* $\mathbb{R}[h] \to H^*(\mathbb{CP}^n)$. *Obviously* $h^{n+1} \to 0$ *if* $h^n \to 0$ *(indeed* $h^n \to [h^n \text{ hyperplanes}] = [pt] \neq 0$*) then each* $p(h^i) \neq 0$ *so it generates our desired* $H^{2i}(\mathbb{CP}^n)$.

Def. *There is something called the* ***tautological line bundle*** *which is* $\{[\vec{v}], \vec{w}\alpha\vec{v}\}$ *where* $\vec{w}$ *is just scalings of* $\vec{v}$. *We see that* $\mathbb{CP}^n = \{[\vec{v}]\}$. *This is as opposed to the better line bundle, which you learn in Algebraic Geometry.*

25.3 More Bundle Stuff

Remark. *Given* M^k *a manifold we have a principal* $GL_k(\mathbb{R})$*-bundle* $\mathit{Frame}(m) \to m$ *and to each* $GL_n\mathbb{R} : X \to X$, *associated* X*-bundle* $X \to (X \times \mathit{Frame}(m))/GL_k\mathbb{R} \to M$.

Ex.

1. $GL_k\mathbb{R} : \mathbb{R}^k \to \mathbb{R}^k$ *then the associated bundle is* TM.
2. $GL_n\mathbb{R} : (\mathbb{R}^k \otimes \cdots \otimes \mathbb{R}^k)^\star$ *where these are antisymmetric tensors and p compositions. The associated bundle is the p-form bundle.*
3. $M \to [1]$ *is the trivial bundle.*
4. $p = k$ *by* $GL_k\mathbb{R} : R \to R$ *by* $M \to [(\det M)^{-1}]$. *This is the k-form bundle or the bundle of volume forms.*
5. $GL_k\mathbb{R} : R \to R$ *where* $M \to [|(\det M)^{-1}|]$ *is the bundle of densities. These are like volume forms but better (since we can integrate without orientation as done previously).*

Remark. *Given* $\varphi : M \xrightarrow{\sim} N$ *we get that* $\mathit{Frame}(M) \xrightarrow{\sim} \mathit{Frame}(N)$ *and* $\mathit{Ass}_X(M) \xrightarrow{\sim} \mathit{Ass}_X(N)$. *Can define twisted p-forms* $GL_k\mathbb{R} : (\mathit{Alt}^p\mathbb{R}^k)^\star \otimes \mathbb{R}$ *with* sign(det) *representation.*

Ex.

1. $GL_k\mathbb{R} : (Sym^2\mathbb{R}^k)^\star$ *by* $(\mathbb{R}^k)\otimes(\mathbb{R}^k)^\star$. *What does a section* $g \in \Gamma(Associated\ Bundle)$ *do?* $g_m(\overrightarrow{v_1}, \overrightarrow{v_2}) \in \mathbb{R}$. *This is indefinite degenerate Riemannian Metrics.*

1. $GL_k(\mathbb{R}) : Sym^2\mathbb{R}^k \cong$ *Symmetric k x k matrices* $\hookleftarrow$ *positive definite matrices with eigenvalue* > 0 *by* $A \cdot M = AMA^T$.

Remark. *What is the use of a Riemannian metric g? Given a Morse function* $f : M \to \mathbb{R}$ *get* $df \in \Gamma(T^\star M)$ *now use* $g : TM \to T^\star M$ *and use g positive definite (nondegenerate) to get* $g^{-1} : T^\star M \to TM$. *Note that positive definite is better than nondegenerate because we can go to submanifolds using positive definite but nondegenerate.*

$M \xrightarrow{df} T^\star M \xrightarrow{g^{-1}} TM$. *This* $g^{-1} \circ f$ *is the gradient of* f, *ie* ∇f.

Ex. $GL_k\mathbb{R} : End(R^k)$ *by* $A \cdot M = AMA^{-1}$. *A submanifold is* $\{M \in End(\mathbb{R}^k) : M^2 = -1\}$. *Note that in this submanifold the eigenvalues of M are* $\pm i$ *in equal numbers (only occurs when* $2 \mid k$*). We have* $k' = k/2$. *Let J be a section of this associated bundle. M has an action J which is rotation by 90 degrees on the tangent space. This makes each real tangent space into a* $\mathbb{C}$*-vector space. This is an* ***almost-complex structure*** *(J the rotation by 90 degrees). The question is that given a manifold could we give it an almost complex structure? A fundamental result is that we can give a Riemannian structure on all manifolds, but what about almost complex? Well we need that* $2 \mid k$.

Theorem 25.2. *Given an almost-*$\mathbb{C}$ *structure J on M ie* $\forall m \in M, J : T_mM \to T_mM$ *where* $J^2 = -1$ *varies smoothly can use to construct an orientation on M. We see that* $T_mM \cong \mathbb{R}^{2k'}$ *and we pick* k' *vectors* $v_1, \ldots, v_{k'}, Jv_1 \ldots, Jv_{k'}$ *has orientation* 1. *We see that this is actually general with respect to how we pick our* v_i *since the set is connected.*

26 Dec 3

26.1 Various Geometries

Def. *An* ***almost complex structures*** *J on a manifold M is a smooth section of* $\{j \in T_mM, j^2 = -I\}$. *Note that this is a bundle over M (not a vector bundle).*

Remark. *We claim that a complex vector space has a natural orientation. To construct a complex vector-space V. Pick a complex basis* $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ *and demand that the orientation of* $\mathcal{O}(\overrightarrow{v_1}, i\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}, i\overrightarrow{v_k}) = 1$. *The basis is unique* *up to $GL(k,\mathbb{C})$, but unlike $GL(k,\mathbb{R})$ this group is connected so the bases are continuously related.f*

Remark. *Recall that the gradient of f is $g^{-1}(df)$ given metric $g : T_mM \to T_m^\star M$. Given a 2-form $\omega \in \Omega^2(M)$ can also define $\omega : TM \to T^\star M$ which is not a metric because antisymmetric.*

Def. *An* ***almost symplectic structure*** *on M is a nondegenerate 2-form ie each $T_mM \to T_m^\star M$ invertible (which implies that the $2 \mid \dim M$). Symplectic Sylvesters law says all symplectic functions on T_mM are equivalent.*

Def. *The* ***symplectic gradient/Hamiltonian vector field*** *of $f : M \to \mathbb{R}$ iw $\omega^{-1}(df)$ where $\omega : TM \to T^\star M$.*

Ex. *Some basic geometries are*

1. *$\mathbb{R}^n$ with usual metric g*
2. *$\mathbb{C}^n$ with usual almost complex form J*
3. *$\mathbb{R}^{2n}$ with usual ω given by $dx_1 \wedge dx_{n+1} + dx_2 \wedge dx_{n+2} \dots dx_n \wedge dx_{2n}$*

If M is locally $\cong$ to one of these, then we can describe it. If it is locally isomorphic to the Euclidean, then if ***complete*** *(ie bounded vector fields have flows, which is implied by compactness), then $M \cong \mathbb{R}^n/\Gamma$ where Γ is a discrete group acting freely.*

If locally isomorphic to symplectic, then by Moser M, ω is a nondegenerate 2 form ie $d\omega = 0$. An example is $T^\star F$ over some field (?).

Remark. *Note that even if M is not compact then we can actually put a complete Riemannian metric on M.*

Ex. *If M is compact oriented and Riemannian, we can use g to put a positive definite inner product on $\Omega^p(M) = \Gamma(M; Alt^pT^\star M)$ which we can use to define $d^\star : \Omega^{p+1}(M) \to \Omega^p(M)$. Compact oriented comes in because*

Def. $\Omega^p(M)_{harmonic} := \{\alpha \in \Omega^p(M) : (dd^\star + d^\star d)\alpha = 0\}$.

Theorem 26.1 (Hodge)**.** *We have that*

$$\begin{tikzcd}
\dots \arrow[r] & \Omega^1(M)_{harmonic} \arrow[r, "0"] \arrow[d, hook] & \Omega^2(M)_{harmonic} \arrow[r, "0"] \arrow[d, hook] & \dots \\
\dots \arrow[r] & \Omega^1(M) \arrow[r] & \Omega^2(M) \arrow[r] & \dots
\end{tikzcd}$$

and this gives an isomorphism on $H^\star$ ie every $c \in H^p(M)$ has a unique harmonic representative. This is incredibly difficult.

Def. *On a* ***Hermitian*** *vector space V, we have $\left\langle \vec{v}, \vec{w} \right\rangle = \overline{\left\langle \vec{w}, \vec{v} \right\rangle}$ and $\overrightarrow{s\vec{v}, t\vec{w}} = \vec{s}\,t \left\langle \vec{v}, \vec{w} \right\rangle$. Note that mathematicians normally put $s\vec{t}$ instead. We take $\Re(\vec{v}, \vec{w}) = g(\vec{v}, \vec{w})$ and $\Im(\left\langle \vec{v}, \vec{w} \right\rangle) = \omega(\vec{v}, \vec{w})$ and so this Hermitian inner product gives us the Riemannian and Symplectic structures. They are related by $g(\vec{v}, J\vec{w}) = \omega(\vec{v}, \vec{w})$.*

Ex. *$\mathbb{CP}^n \cong U^{n+1} \cdot e_{11}$ where $\cdot$ is action by conjugation. The complex structure is given by $\mathbb{C}\vec{v} \to \vec{v} \cdot \vec{v}^T / |\vec{v}|^2$. So every tangent space is Hermitian.*

Def. *A* ***Kahler manifold*** *is complex with choice of Hermitian structure on TM.*

Theorem 26.2 (Hard Lefschetz)**.** *If M is compact and Kahler ie has $\omega \in \Omega^2(M)$, then*

1. *The map $H^{n-i}(M) \xrightarrow{\wedge[\omega]^i} H^{n+i}(M)$ is an isomorphism. Note that these two vector spaces are isomorphic since M is compact and Kahler implies oriented and we just use Poincaré duality.*
2. *$[\wedge\omega, (\wedge\omega)^\star] = (p-n)\mathbb{1}$ on H^p.*

This means that the Betti numbers form a palindrome that is increasing then decreasing (it's a mountain) ie something like $1, 2, 3, 4, 5, 4, 3, 2, 1$.

Remark. *There was a push to try to prove that Symplectic manifolds are Kahler. But someone found a manifold with Betti Numbers that aren't a mountain, which disproved the assertion.*

Theorem 26.3. *If $A, B \hookrightarrow M$ are all complex (so oriented) and $A \pitchfork B$, then the orientations on all are compatible. The special case is that if $A \pitchfork B$ is a set of points, then all points have positive orientation.*

Ex. *Let $V = \mathbb{C}^3$. Then $\{(p, \ell) : \ell \ni p, [\star 00]\} \hookrightarrow \mathbb{P}V \times \mathbb{P}V^\star$, where $\mathbb{P}V$ are the points and $\mathbb{P}V^\star$ are the liens. This is just projective geometry. note that given p we can find the unique line ℓ if $p \neq [\star 00]$, but we must include the ℓ in our formulation since if $p = [\star 00]$ we can pick the line. This means that mostly $\mathbb{P}V \hookrightarrow \mathbb{P}V$ but sometimes we must take it with the dual.*

27 Dec 5

27.1 Abstract Manifolds

Def. *An* ***abstract manifold*** M *is a union of open subsets* $U_i, i \in I$ *s.t. for* $V_i \subseteq \mathbb{R}^k$ *there is a bijection* $\varphi_i : V_i \to U_i$ *and for all* $i, j \in I$ *we have that* $\mathbb{R}^k \supseteq V_i \supseteq \varphi_i^{-1}(U_i \cap U_j) \xrightarrow{\varphi_i} U_i \cap U_j \xleftarrow{\varphi_j} \varphi_j^{-1}(U_i \cap U_j) \subseteq V_j \subseteq \mathbb{R}^k$.

Ex. *For submanifolds of* $\mathbb{R}^n$*, at each* $m \in M$ *ask hat are of the* $\binom{n}{k}$ *projections* π *be locally invertible and smooth from* $\mathbb{R}^k \to \mathbb{R}^n$.

Def. *A* ***tangent space*** *to an abstract manifold is* $T_mM = \{\overrightarrow{v} \in T_{|varphi_i^{-1}(m)}\mathbb{R}^k\}$ *for each* $U_i \ni m$*. This is also equal to* $\{\gamma : (-1, 1) \to M\}/\sim$ *where the equivalence relation is if the have the same derivative at* 0*. Concretely, this is* $\mathrm{im}(T(\pi^{-1}))$ *and shown in our example. In algebraic geometry, the* ***Zariski*** *definition is the dual to the quotient of the ideal of smooth functions on open sets at* m *vanishing at* m *with products of two such functions.*

Def. ***Tangent bundle*** *is* TM *is defined by* $M \times \mathbb{R}^k$*. The* ***derivative*** *is defined for* $f : M \to N$ *by* $Tf : TM \to TN$ *as a map* $T_mM \xrightarrow{linear} T_{f(m)}N$*.*

Theorem 27.1 (Inverse function thm)**.** *If* $f : M \to N$ *is smooth and* $Tf : T_mM \to T_{f(n)}N$ *is invertible then*

1. $\exists U \subseteq M$, $U \ni m$ *s.t.* $f : U \to f(U)$ *is a diffeomorphism*
2. *or there exists charts* U, U' *around* $m, f(m)$ *that are diffeomorphic to the same* $V \subseteq \mathbb{R}^k$*.*

Theorem 27.2 (Local Immersion Thm)**.** *If* $f : M \to N$, T_mf *is* $1:1$ *then there exist coordinates* $\varphi : V \xrightarrow{\sim} U \subseteq M$ *around* m *and* $\varphi' : V' \to U'$ *around* $f(m)$ *s.t. the map is the canonical immersion ie something like* $\begin{pmatrix} I_n & 0 \end{pmatrix}$*. In particular* $f : U \to U'$ *is an immersion.*

Ex.

1. *A bad example is* $\mathbb{R} \to \mathbb{R}^2$ *given by looking like the figure* 8 *(except not completed* 8 *where the arrows are) and note that this is bad.*
2. *Even when injective if we make a loop s.t. as* $x \to \infty$ *it approaches the original line*
3. *For the irrational flow on* T^2*, we note that while the image is dense in* T^2*. It is clearly not onto although it is dense (topologically onto), and this suffers from the same problem as above (it is not proper).*

Def. *$f : M \to N$ is an **embedding** if proper injective immersion. This implies* $\mathrm{im}(f)$ *is a submanifold.*

Def. *$m \in M$ is a **regular point** if $T_m f$ is onto $n \in N$ is a regular value if $f^{-1}(n)$ is regular points. $f : M \to N$ is a **submersion** if all values are regular.*

Proof of local immersion thereom. In charts, we have $f : A \to B$, $T_0 f$ is $1 : 1$. We have $f_+ : A \times (\mathrm{im}\, T_0 f)^\perp \to \mathbb{R}^k$ by $(a, \vec{v}) \to f(a) + \vec{v}$. Applying inverse function theorem to f_+. $\square$

Theorem 27.3 (Local Submersion Thm)**.** *If $f : M \to N$, $T_m f$ is onto, then f is a local submersion. $\exists$ coordinates s.t. f looks like $\mathbb{R}^j \twoheadrightarrow \mathbb{R}^k$.*

Def. *$f : M \to \mathbb{R}$ fails to be a submersion at $p \in \mathbb{R}$ (a critical point) is **Morse** if its Hessian is nondegenerate.*

Ex. *$f : \mathbb{R}^n \to R$ by $(x_1, x_2, \ldots, x_n) \to \sum \pm x_i^2$.*

Theorem 27.4 (Morse Lemma)**.** *Around Morse Critical points, there exist coordinates looking like the basic example.*

Theorem 27.5 (Preimage Theorem)**.** *If $f : M \to N$ has regular value $p \in N$, then $f^{-1}(P)$ is a* $\dim M - \dim N$*-manifold.*

Def. *The **normal bundle** to $M \hookrightarrow L$, $N_M L = \{(\ell, \vec{v}) \in TL \mid \ell \in M, \vec{v} \perp T_m M\}$. Algebraic geometry would say $\vec{v} \in T_m L / T_m M$ instead because $\perp$ is gross.*

Theorem 27.6 (Tubular Nbhd Theorem)**.** *If $M \hookrightarrow N$ is a closed submanifold then there exists an open set $U : M \subseteq U \subseteq L$ s.t. $U \xrightarrow{\sim} N_M L$.*

Theorem 27.7 (Sard's Theorem)**.** *Almost every value is regular.*

Def. *$A, B \hookrightarrow M$ are **transverse** if $T_p A + T_p B = T_p M$ for all $p \in A \cap B$.*

Def. *A manifold with boundary is defined similarly to a manifold except with half space.*

Def. *Orientation on a real vector spaces $\mathcal{O} : \{bases\} \to \{\pm 1\}$ given by* $\mathcal{O}(B)\mathcal{O}(M \cdot B) = \mathrm{sign}(\det M)$*.*

28 Dec 10

28.1 Review of Forms

Def. *$\Omega^p(M)$ is smooth p-forms ie antisymmetric multilinear maps from $(m, T_m M) \to \mathbb{R}$ and this is $\cong \Gamma(M; \wedge^p T^\star M)$ which are p-th exterior powers on the cotangent bundle.*

We have operations on a $f : M \to N$ with $\Omega^\star(M) \xleftarrow{f^\star} \Omega^p(N)$ defined as the pullback, $\wedge : \Omega^p(M) \times \Omega^q(M) \xrightarrow{bilinear} \Omega^{p+q}(M)$, and $d : \Omega^p(M) \to \Omega^{p+1}(M)$.

For the algebra, we have that $d^2 = 0, (f \circ g)^\star = g^\star \circ f^\star, f^\star(\alpha \wedge \beta) = f^\star\alpha \wedge f^\star\beta, \alpha\wedge\beta = (-1)^{\deg\alpha\deg\beta}\beta\wedge\alpha, (\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma), f^\star(d\alpha) = df^\star(\alpha), d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge d\beta$.

Def. *For any **Complex** $\xrightarrow{d} A_1 \xrightarrow{d} A-2\dots$ with $d^2 = 0$, we define $H^i(A^\cdot) = (\ker A_i \xrightarrow{d} A_{i+1})/\operatorname{im}(A_{i-1} \to A_i)$. This forms the idea of SES : $0 \to A \to B \to C \to 0$ which implies LES.*

Ex. ***Mayer-Vietoris*** *is defined as for $M = U \cup V$ we have*

$$0 \to \Omega(M) \xrightarrow{M \leftarrow U \cup_{disjoint} V)^\star} \Omega(U) \oplus \Omega(V) \xrightarrow{difference} \Omega(U \cap V) \to 0$$

Def. *A **good cover** of M is a collection of open sets s.t. any finite nontrivial intersection is $\emptyset$ or contractible.*

Theorem 28.1. *Based on Riemannian Geometry (geodesic nbhds), we have that there always exists a good cover on M.*

Remark. *In Riemannian Geometry, we note that for larger dimensions (ie > 4) because of magic with the fundamental group, this becomes trivial. However, there are still people working on 3 and 4 manifolds (since 2 is trivial).*

Lemma (Poincaré). $H^\star(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \star = 0 \\ 0 & \star > 0 \end{cases}$

Def. *Orientation on M is $\int_M$ compactly supported top forms on M.*

Def. *Boundary Orientation*

Theorem 28.2 (Stokes). $\int_M d\alpha = \int_{\partial M} \alpha$ *for oriented manifolds with boundary.*

Theorem 28.3 (Poincaré Pairing). $\Omega^k(M) \times \Omega_c^{\dim M - k} \xrightarrow{\int \wedge} \mathbb{R}$

Theorem 28.4 (Poincaré Duality). *This descends to a perfect pairing* $H^k(M) \times H_c^{\dim M-k}(M) \to \mathbb{R}$

Theorem 28.5. *If* $f_0, f_1 : M \to N$ *are homotopic then* $f_0^\star = f_1^\star$ *as* $H^\star(N) \to H^\star(M)$.

Def. *If* $M \hookrightarrow N$ *both compact oriented,* $[M] \in H^{\operatorname{codim}_N M}(N)$ *is the* ***Poincaré dual****. We have that* $(H^e(N) \xrightarrow{\iota^\star} H^e(M) \xrightarrow{\int_M} \mathbb{R}) \in (H^e(N))^\star \cong H^c(N)$ *where* $e = \dim M$ *and* $c = \operatorname{codim}_N M$.

Ex. $\{0\} \hookrightarrow \mathbb{R}$ *with* $[\{0\}]$ *are bump forms on* $\mathbb{R}$ *ie compactly supported non-negative* $\int = 1$. *More generally* $\{0\} \hookrightarrow V$ *on an oriented vector space. Even more generally, we have* $M \hookrightarrow V$ *on the oriented vector bundle is a* ***Thom Form****.*

Def. *If* $M \leftarrow V$ *oriented vector bundle a* ***Thom form*** α *has*

(i) *has degree* $= \operatorname{codim} M = \dim \mathit{Fibers}$

(ii) *is a bump form on every fiber*

(iii) *closed*

An so $\alpha \in \Omega^{\dim \mathit{Fiber}}(V)$.

If $M \hookrightarrow P$ *are both oriented, then* $M \hookrightarrow N_M P \cong$ *tubular nbhd extended by 0* $\hookrightarrow P$.

Theorem 28.6. $[M]$ *is the class of extension by* 0. *Thom form on* $N_M P$.

28.2 Intersection Numbers

Def. $\chi(M) = I(\triangle, \triangle)$ M *is compact oriented*

Theorem 28.7 (Poincaré Hopf). *This is equal to* $\sum_i (-1)^i \dim H^i(M)$.

Def. ***Lefschetz Number*** *of* $f : M \to N$ $I(\triangle, \operatorname{graph} f)$ *for Lefschetz functions* f.

Theorem 28.8. *This is equal to* $\sum_i (-1)^i tr(f : H^i(M) \to H^i(M))$.