Math 6510 Algebraic Topology

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Spring 2021

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1 Introduction

Classical topology deals with homeomorphisms. Think of mug and donut example.

Homotopy equivalence is much different from standard geometry and stuff. For example $\{v\} \cong D^n$ for any n.

Def. *$f, g : X \to Y$ (topological spaces) are **homotopic** if $\exists$ map $H : X \times I \to Y$ s.t. $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. We denote this by $f \sim g$.*

Ex.

- *Let $X = D^n$ and $Y = D^n$. $f : X \to Y$ is identity and $g : X \to Y$ is 0. $H : X \times I \to Y$ would be $H(x, t) = (1 - t)x$.*
- *$X = S^1, Y = \mathbb{R}^2 \setminus \{0\}$. f is identity and g is a constant map to some $x \in Y$. Not homotopic since circle can't go to a point.*

Def. *X and Y are **homotopy equivalent** if there exists a map from $f : X \to Y$ and $g : Y \to X$ s.t. $f \circ g \sim \mathbb{1}_Y$ and $g \circ f \sim \mathbb{1}_X$. We denote by $X \cong Y$.*

Ex. *If X and Y are **homeomorphic** then they are homotopic.*

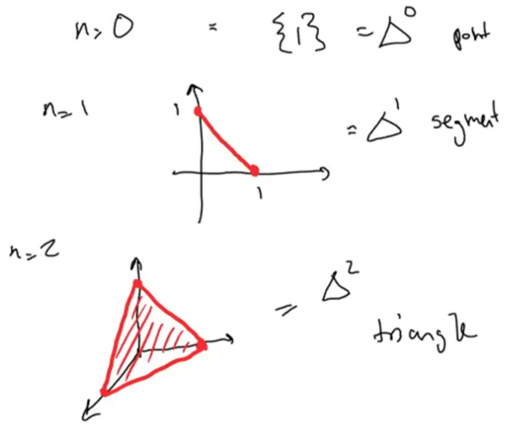
Ex. *$* \cong D^n$. $f : D^n \to *$. $g : * \to D^n$ goes to 0. $fg\mathbb{1}_*$. $gf : D^n \to \{0\}$. Can see that this is homotopic to identity by scaling.*

2 Homology

2.1 Simplicial Homology

We can construct larger topological spaces by gluing together existing topological spaces.

Def. *A **simplex** $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1\}$. $n = 0$ is a single point. $n = 1$ is a segment on the plane. $n = 2$ is a triangle in three dimensions.*



Def. *An **abstract simplicial complex** consists of the following data:*

- *a set of vertices V*
- *a set of simplices S, where a simplex is a nonempty finite subset of V.*

S has the property that if $X \in S$ and $T \subseteq X$ then $T \in S$.

Ex. *$V = [5]$ and subsets of $\{3,4,5\}, \{1,2\}, \{2,3\}, \{1,3\}$.*

In an abstract simplicial complex, a simplex is uniquely determined by the vertices.

Ex. *We can model a circle $V = [3]$ by $1, 2, 3, 12, 23, 13$. But $V = [2]$ can't have a circle.*

Def. *A **chain complex** C_* is a sequence of abelian groups $\{C_n \mid n \in \mathbb{Z}\}$ together with maps $d_n : C_n \to C_{n-1}$ s.t. $\forall n\ d_n \circ d_{n+1} = 0$. The d_n are called **boundary maps**.*

Def. *Given a simplicial complex K, pick **any** ordering on its vertices V. C_n is the free abelian group generated by n-simplices in K ($(n+1)$ size subsets). $d_n : C_n \to C_{n-1}$ by $d_n(\{x_0, \ldots, x_n\}) = \sum_{i=0}^{n} (-1)^i [\{x_0, \ldots, \widehat{x_i}, \ldots, x_n\}]$ where the x_i are in order.*

Remark. *It suffices to pick a partial ordering s.t. all simplices are totally ordered.*

We note that $d_n \circ d_{n+1} = 0$. This is because for any simplex we have that

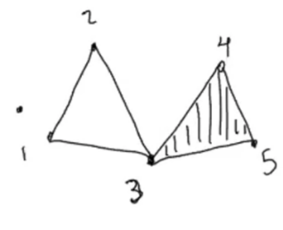
$$\begin{aligned}
d_n \circ d_{n+1}(\{x_0, \dots, x_{n+1}\}) &= d_n \sum_{i=0}^{n+1} (-1)^i [\{x_0, \dots, \widehat{x_i}, \dots, x_{n+1}\}] \\
&= \sum_{i=0}^{n+1} (-1)^i (\sum_{j=0}^{i-1} (-1)^j [\{x_0, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_{n+1}\}] \\
&+ \sum_{j=i}^{n} (-1)^j [\{x_0, \dots, \widehat{x_i}, \dots, x_{j+1}\}]) \\
&= \sum_{0 \le i < j \le n} \sum ((-1)^{i+j-1} + (-1)^{i+j}) [\{x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}\}] \\
&= 0
\end{aligned}$$

Def. *Given a chain complex C_*, the n-th* ***homology*** *of C_*, $H_n C_*$ is defined as $H_n C_* := \ker d_n / \operatorname{im} d_{n+1}$. Note that since $d_n \circ d_{n+1} = 0$, this is well defined.*

Ex.

- *Given a single point $\{*\}$ then $C_n = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases}$. Therefore $H_n C_* = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases}$ (note that $d_0 : C_0 \to C_{-1}$ must have a kernel of the entire thing and $d_1 : C_1 \to C_0$ has an image of 0.*

- *Suppose we have the example*



$C_n = 0$ if $n \neq 0, 1, 2$. $C_0 = \mathbb{Z}^5$ (5 vertices). $C_1 = \mathbb{Z}^6$ (6 edges). $C_2 = \mathbb{Z}$ (1 face). $d_0 = 0$ since we are mapping to d_{-1}.

$d_2 : C_2 \to C_1$ takes $[\{3,4,5\}] \to [\{4,5\}] - [\{3,5\}] + [\{3,4\}]$. This is nonzero, so d_2 is injective. Therefore $\ker d_2 = 0 \implies H_2 C_* = 0$.

For H_0, we note that $\ker d_0 = \mathbb{Z}^5$. *d_1 maps an edges $[\{x,y\}]$ to $[\{y\}] - [\{x\}]$. Any two points in the same component differ by an element in the image of d_1. For example, $d_1(-[\{1,3\}] - [\{3,4\}]) = [\{1\}] - [\{4\}]$. This means that $H_0C_* = \mathbb{Z}$. We can get this by adding up the coefficients of the 0-simplices. This is because all elements of c_0 are $a_1[\{1\}] + a_2[\{2\}] + \cdots + a_5[\{5\}]$ and we can collapse these down to remove $[\{1\}]$.*

Now, we do H_1. Examine $d_1 : C_1 \to C_0$. The kernel of d_1 is generated by $[\{1,2\}] + [\{2,3\}] + [-\{1,3\}]$ and $[\{3,4\}] + [\{4,5\}] + [-\{3,5\}]$. Therefore the kernel is $\mathbb{Z}^2$. The image of d_2 is $d_2[\{3,4,5\}] = [\{3,4\}] - [\{3,5\}] + [\{4,5\}]$ (which is the second part that is killed) $H_1C_* \cong \mathbb{Z}$.

Remark. *We can see why the d_n are called a boundary map. In particular, for the second example above, draw arrows for the edges generated by d_2, with negative edges being reversed.*

We also see that Homology measures in "holes". In particular, since $H_1C_* \cong \mathbb{Z}$ then there is a "hole" is the triangle $\{1,2,3\}$.

2.2 Axioms of Homology

The above is homotopy equivalent, which is something we want to prove. To do this, we develop homology from an axiomatic viewpoint, as opposed to the computational viewpoint described above.

What do we want from Homology?

- Independent of simplicial structure.
- Homotopy Invariant. Defined on maps. In particular for a $f : X \to Y$ we want $f_* : H_*X \to H_*Y$. If $f, g : X \to Y$ and $f \sim g$, then $f_* = g_*$.
- Meaningful.

Def. *A* ***pair of space*** *(X, Y) is a space X and a subspace $Y \subseteq X$ s.t. there is an open neighborhood of Y that is homotopic to Y.*

Ex. *The sin curve $(x, \sin\frac{1}{x})$ and the y axis inside of $\mathbb{R}^2$ is a non example.*

Ex. $S^n \subseteq D^{n+1}$.

Ex. *Any subcomplex in a simplicial complex.*

Def. *A* ***map*** *of pairs of spaces $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ s.t. $f(A) \subseteq B$. A* ***homotopy*** *of spaces f, g is a map of spaces $H : (X \times I, A \times I) \to (Y, B)$.*

Def (Weintrab Definition)**.** *A* ***homology theory*** *associates to each pair of spaces (X, A) a sequence of abelian homology groups $\{H_i(X, A)\}_{i\in\mathbb{Z}}$ and a sequence of homomorphisms $\{\partial_i : H_i(X, A) \to H_i(A, \emptyset)\}_{i\in\mathbb{Z}}$. It assigns to each map $f : (X, A) \to (Y, B)$ a sequence of homomorphisms $\{f_i : H_i(X, A) \to H_i(Y < B)\}$ satisfying*

(A1) If $f : (X, A) \to (X, A)$ is identity, then f_i is identity.

(A2) If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ one maps with $h = g \circ f$, then $h_i = g_i \circ f_i$.

(A3) The following diagram commutes

$$\begin{array}{ccc} H_i(X, A) & \xrightarrow{f_i} & H_i(Y, B) \\ \big\downarrow \partial_i & & \big\downarrow \partial_i \\ H_i(A, \emptyset) & \xrightarrow{(f|_A)_{i-1}} & H_{i-1}(B, \emptyset) \end{array}$$

where $f|_A : (A, \emptyset) \to (B, \emptyset)$.

(A4) We also write $H_i(A) := H_i(A, \emptyset)$ for brevity. The following sequence is exact

$$\cdots \to H_i(A) \to H_i(X) \to H_i(X, A) \xrightarrow{\partial_i} H_{i-1}(A) \to H_{i-1}(X) \to H_{i-1}(X, A) \to \dots$$

(A5) If f and g are homotopic, then $f_i = g_i$.

(A6) $U \subseteq A$ s.t. $\overline{U} \subseteq A^\circ$ and $f : (X - U, A - U) \to (X, A)$ then $f_i : H_i(X - U, A - U) \to H_i(X, A)$ is an isomorphism.

(A7) $H_i(\{a\}) = 0$ if $i \neq 0$.

Def. *A composition of abelian groups*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

*is **exact** at B if $\ker g = \operatorname{im} f$. A sequence of homomorphisms of abelian groups*

$$\cdots \to A_{i+1} \to A_i \to A_{i-1} \to \dots$$

is exact if it is exact at A_i for all $i \in \mathbb{Z}$. A finite sequence

$$A_1 \to \cdots \to A_n$$

is exact if it is exact at each map.

Ex. *$A \xrightarrow{f} B \to 0$ is exact at B iff f is surjective. $0 \to A \xrightarrow{f} B$ is exact at A iff f is injective. $0 \to A \xrightarrow{f} B \to 0$ is exact at A and B iff f is an isomorphism.*

Ex. *A **short exact sequence** is a sequence*

$$0 \to A \to B \to C \to 0$$

which is exact everywhere.

Ex. *A **long exact sequence** is a sequence*

$$\cdots \to A_{i+1} \to A_i \to A_{i-1} \to \dots$$

which is exact everywhere.

Lemma. *If (X, A) s.t. the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then $H_i(X, A) = 0$ and $H_i(\emptyset) = H_i(X, X) = 0$.*

Proof. Use LES.

$$\cdots \to H_i(A) \xrightarrow{\cong} H_i(X) \to H_i(X, A) \to H_{i-1}(A) \xrightarrow{\cong} H_{i-1}(X) \to \dots$$

Note that $H_i(A) = H_i(X)$ by homotopy equivalence. So the kernel from $H_i(A) \to H_i(X)$ has kernel 0 and full image. This must imply that $H_i(X, A) = 0$. $\square$

We now examine how homotopy equivalences relate homology.

Lemma. *Let* $(X, A) \xrightarrow{f} (Y, B)$ *be a homotopy equivalence. Then* f_i *is an isomorphism for all* i.

Proof. We can proof using functoriality (1, 2) and homotopys invariance (5) $\square$

But, we can do better.

Prop 2.1. *Let* $(X, A) \xrightarrow{f} (Y, B)$ *and suppose that* $f : X \to Y$ *and* $f|_A : A \to B$ *homotopy equivalences.* f_i *is an isomorphism for all* i.

Proof. Axiom 4 shows that there is a LES

$$\begin{tikzcd}
\dots \arrow[r] & H_i(A) \arrow[r] \arrow[d, "f_i"] & H_i(X) \arrow[r] \arrow[d, "f_i"] & H_i(X, A) \arrow[r] \arrow[d, "f_i"] & H_{i-1}(A) \arrow[r] \arrow[d, "f_{i-1}"] & H_{i-1}(X) \arrow[r] \arrow[d, "f_{i-1}"] & \dots \\
\dots \arrow[r] & H_i(B) \arrow[r] & H_i(Y) \arrow[r] & H_i(Y, B) \arrow[r] & H_{i-1}(B) \arrow[r] & H_{i-1}(Y) \arrow[r] & \dots
\end{tikzcd}$$

We see that the first two downward maps and the last two downward maps are isomorphisms, and by the five lemma, the middle down arrow is also an isomorphism since this is a LES. $\square$

Remark. *The five lemma is a very annoying diagram chase. But, the result is pretty powerful, and we're gonna use it often.*

Remark. *This is strictly stronger because there are homotopies on* X *alone which break the structure of* A*. For example, consider* X *being a line segment and* A *being the two endpoints.* X *is homotopic to a rectangle, but clearly we can break* A *by changing the way we collapse the rectangle back into a line.*

Lemma. *Suppose* $j : A \hookrightarrow X$ *and there exists a* $r : X \to A$ *s.t.* $r \circ j = \mathbb{1}_A$*. We call this a* ***retraction****. Then* j_* *is an injection and*

$$H_i(X) \cong H_i(A) \oplus H_i(X, A) \text{ and } H_i(X, A) \cong \ker r_i$$

Ex. *An example is a figure* 8*, where we retract one half into a point. This is topologically nontrivial.*

Proof. By axiom 4, we have that

$$\begin{tikzcd}
H_i(A) \arrow[r, "j_i"] \arrow[dr, "="'] & H_i(X) \arrow[r] \arrow[d, "r_i"] & H_i(X,A) \arrow[r] & H_{i-1}(A) \arrow[r] & \dots \\
& H_i(A) & & &
\end{tikzcd}$$

We see that this implies that j_i is an injection and r_i is a surjection. Therefore $\ker j_i$ is 0, to the map into $H_i(A)$ is the zero map. This makes that we have a SES $\xrightarrow{0} H_i(A) \to H_i(X) \to H_i(X,A) \xrightarrow{0} H_{i-1}(A)$. Therefore $H_i(X) \to H_i(X,A)$ is a surjection and $H_i(A) \to H_i(X) \to H_i(X,A)$ forms a SES.

Whenever a SES has a splitting, the right hand group is the kernel of the splitting (exercise?). We have that $H_i(X,A) \cong \ker r_i$ and $H_i(X) \cong H_i(A) \oplus H_i(X,A)$. □

Remark. *A split SES is a SES* $0 \to A \to B \to C \to 0$ *where there are backwards maps (either from* $B \to A$ *or* $C \to B$*) which compose to form the identity with the forward map.*

Ex. *An example is* $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$ *where the first map is* $a \to (a,a)$*. Then clearly there is an inverse* $(a,b) \to a$ *so this split on the left. Not all SES splits, for example* $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ *where the first map is* $1 \to 2$ *and the second is* $1 \to 1$*.*

Remark. *Homotopy has the same axioms as homology except for axiom 6, the closure rule. This is the reason why homology is computable, but homotopy is not.*

Ex. $X = [0,1], A = \{0\}$ *and* $U = A$*. Then* $\overline{U} \not\subseteq A^\circ$*. We notice that* $j : A \hookrightarrow X$ *is a homotopy equivalence, so* $H_i(X,A) = 0$*. But* $H_i(X - U, A - U) = H_i([0,1], \emptyset) \cong H_i(\{*\})$ *which is not* 0*.*

Def. *A is a **strong deformation retract** of W if $A \subseteq W$ and there exists a retraction $r : W \to A$ and a $j : A \hookrightarrow W$ s.t. $j \circ r \sim \mathbb{1}_W$ via a homotopy which is uniformly the identity on A.*

This implies that A/A is a strong deformation retraction of W/A, so retractions hold.

Theorem 2.1. *Let A be a nonempty closed subset of X and suppose that A is a strong deformation retract of an open neighborhood in X. Then $H_*(X,A) \cong H_*(X/A, [A])$ where A is a point in X/A.*

Remark. *A point is always a retract of a nonempty space.* $H_i(X) \cong H_i(\{*\}) \oplus H_i(X, \{*\})$*. Recall that* $H_i(\{*\}) = \mathbb{Z}$ *only for dim* 0*, otherwise it is* 0*. Then* $H_i(X, \{*\})$ *is called the reduced homology* $\widetilde{H}_i(X)$*.*

Proof. Let $A \subseteq W \subseteq X$ where A is closed in X and W is open in X. We have that

$$H_i(X, A) \cong H_i(X, W) \cong H_i(X - A, W - A) \cong H_i(X/A - A/A, W/A - A/A)$$
$$\cong H_i(X/A, W/A) \cong H_i(X/A, A/A)$$

where the congruences follow by

- Prop 2.1
- Axiom 6
- Induced by Homeomorphism
- Axiom 6
- Prop 2.1

We need strong deformation retraction to keep the homotopies homotopies after quotienting by A. □

We can now use our homology to compute the homology of a sphere.

Theorem 2.2. $\widetilde{H_i}(S^n) \cong \begin{cases} H_i(pt) & i = n \\ 0 & otherwise \end{cases}$*. In particular, it follows that* $H_i(S^n) = \mathbb{Z}$ *iff* 0 *or* n*, and* 0 *otherwise.*

Remark. *This is why reduced homology is nicer. We don't have to worry about* 1 *connected component and this gives us nice theorems that don't hold for homology.*

Proof. We do this by induction. It is obvious for $n = 1$.

Consider the pair (D^{n+1}, S^n). D^{n+1} is homotopy equivalent to a point.

$$\cdots \to H_{i+1}(D^{n+1}) \to H_{i+1}(D^{n+1}, S^n) \to H_i(S^n) \to H_i(D^{n+1}) \to \dots$$

If $i > 0$ then $H_i(D^{n+1}) = 0$ and this implies that $H_{i+1}(D^{n+1}, S^n) \cong H_i(S^n)$. By our induction hypothesis, this implies that $H_i(S^n) \cong 0$ if $i \neq n$, and this implies that $H_{i+1}(D^{n+1}, S^n) \cong 0$ if $i \neq n$. We get that $H_{i+1}(D^{n+1}, S^n) \cong H_{i+1}(D^{n+1}/S^n, \{*\}) \cong H_{i+1}(S^{n+1}, \{*\})$.

Now, examine $n = 1$. We examine (D^1, S^0) and this gives us a LES $H - i(D^1) \to H_i(D^1, S^0) \to H_{i-1}(S^0) \to H_{i-1}(D^1) \to H_{i-1}(D^1, S^0) \to \dots$. Note that D^1 is contractible so $H_*(D^1) = H_*(pt)$ so $H_*(S^0) = H_*(pt) \oplus H_*(pt)$ if r $i \neq 0$. This is 0. If $i > 1$ or < 0 we have that

$$H_i(D^1) \to H_i(D^1, S^)) \to H_{i-1}(S^0) \implies H_i(D^1, S^0) = 0$$

If $i = 1$ then

$$H_1(D^1) \to H_1(D^1, S^0) \to H_0(S^0) \to H_0(D^1) \to H_0(D^1, S^0) \to H_{-1}(S^0)$$

We see that $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(D^1) = \mathbb{Z}$ and $H_{-1} = 0$. We examine the map from $H_0(S^0) \to H_0(D^1)$. If it is $(a, b) \to a + b$, it is surjective so $H_0(D^1, S^0) = 0$ and $H_1(D^1, S^0) = \ker \cong \mathbb{Z}$ and we're done. But if this map is 0, then $H_1(D^1, S^)) \cong H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(D^1, S^0) \cong H_0(D^1) \cong \mathbb{Z}$. We need to determine this map in order to compute the homology.

How do we find $H_0(S^0) \to H_0(D^1)$. The map $S^0 \to D^1$ is the inclusion map and we can compose this with a point $pt \to S^0 \to D^1$ where the point goes to 1. Note that this map from $pt \to D^1$ is a homotopy equivalence, and homotopy equivalence means that the maps are isomorphisms. So the 0 homologies are $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. The first map is $a \to (a, 0)$ since it's a map of disjoint unions (inclusion), so therefore the second map must be map $(a, 0) \to a$, so it must $(a, b) \to a + b$.

So we have that $H_*(D^1, S^0) = \begin{cases} \mathbb{Z} & * = 1 \\ 0 & \text{ow} \end{cases}$. In particular, we see that $H_*(D^1, S^0) = \widetilde{H}_*(D^1/S^0) \cong \widetilde{H}_*(S^1)$ by quotienting and it's a deformation retraction. This gives us the homology of S^1. $\square$

We continue with more examples.

Def. *We recall the definition of a* ***cone*** $CX \cong X \times I/(x,0) \sim (x',0)$.

Note that CX is homotopy equivalent ot a point with a map $H((x,s),t) = (x, st)$.

Def. *We define the* ***suspension*** *of* X *given by* $\Sigma X = CX/X$ *through the inclusion map* $x \to (x,1)$. *Think of this as* $CX \cup CX$ *unioned at* X *between them.*

Ex. $S^n \cong \Sigma S^{n-1}$.

Theorem 2.3. *For all spaces* X *there is an isomorphism* $\Sigma : \widetilde{H}_{i+1}(\Sigma X) \to \widetilde{H}_i(X)$. *This is natural in the sense that for all* $f : X \to Y$ *the following diagram commutes*

$$\begin{tikzcd} \widetilde{H}_{i+1}(\Sigma X) \arrow[r, "\Sigma"] \arrow[d, "f_*"] & \widetilde{H}_i(X) \arrow[d, "f_*"] \\ \widetilde{H}_{i+1}(\Sigma Y) \arrow[r, "\Sigma"] & \widetilde{H}_i(Y) \end{tikzcd}$$

Proof. Inside CX, X has a nice neighborhood (a little bit above) so $\widetilde{H}_*(\Sigma X) = \widetilde{H}_*(CX/X) \cong H_*(CX/X)$. We have a LES

$$\cdots \to H_i(CX) \to H_i(CX, X) \to H_{i-1}(X) \to H_{i-1}(CX) \to \dots$$

Let $x_0 \in X$. Then $g : (x,s) \to x_0$ is a map. This induces a splitting of the map $H_i(X) \to H_i(CX)$ and we can map $H_i(CX) \xrightarrow{g_i} H_i(X)$. Therefore, we have an isomorphism by taking g_i first then mapping back; this is possible because CX is contractible. This implies that $H_i(X) \to H_i(CX)$ is surjective, so $H_i(CX) \to H_i(CX, X)$ is 0.

We therefore have short exact sequences $0 \to H_i(CX, X) \to H_{i-1}(X) \to H_{i-1}(CX) \to 0$. When $i - 1 \neq 0$, then $H_{i-1}(CX) = 0$ which implies that $H_i(CX < X) \cong H_{i-1}(X)$. When $i - 1 = 0$, then we have that $0 \to H_1(CX, X) \to H_0(X) \to H_0(CX) \to 0$. We see that $H_1(CX, X) \cong \ker(H - 0(X) \to H_0(pt)) \cong \widetilde{H}_0(X)$.

We proved this because we wanted to show that the map $H_i(CX, X) \to H_{i-1}(X)$ is Σ. $\square$

Now we examine the torus. Recall that a torus can be realized as a square with opposite sides glued together. Let the horizontol lines form a green circle and the verticle sides form a red circle.

Let A be the green circle and X the figure 8 formed by joining together the red and green circles at the point p. We have the LES

$$\to H_i(A) \to H_i(X) \to H_i(X, A) \to H_{i-1}(A) \to \dots$$

Note that $H_i(X, A) = \widetilde{H}_i(X/A) \cong \widetilde{H}_i(red) \cong 0$ for $i \neq 1$. At $0, 1$ we have

$$0 \cong H_2(X, A) \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A) \cong 0$$

Note that $H_*(A) \to H_*(X)$ is injective since this has an inverse induced by collapsing the red circle. We have a SES $0 \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$ and $H_1(X) \cong \mathbb{Z}^2$.

If $T =$ torus, then we want to learn about the homology of T from the homology of X. Note that (T, X) is a good pair and $H_i(T, X) \cong \widetilde{H}_i(T/X)$. Note that $T/X \cong S^2$, so $\widetilde{H}_i(T/X) \cong \begin{cases} \mathbb{Z} & i = 2 \\ 0 & \text{ow} \end{cases}$. We have a LES

$$H_{i+1}(X) \to H_{i+1}(T, X) \to H_i(X) \to H_i(T) \to H_i(T, X) \to \dots$$

if $i \neq 1, 2$ we get that $0 \to H_i(X) \to H_i(T)$ so $H_i(X) \cong H_i(T)$. The only interesting case is

$$0 = H_2(X) \to H_2(T) \to H_2(T/X) \to H_1(X) \to H_1(T) \to 0$$

Note that $H_2(T) \to H_2(T/X)$ is injective and $H_1(X) \to H_1(T)$ is surjective. $H_2(T/X) \to H_1(X)$ is a map from $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. If it is 0 then $H_2(T) \cong \mathbb{Z}$ $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ or it is not 0 and $H_2(T) \cong 0$ and $H_1(T) \cong \mathbb{Z}$.

We know that that $H_1(X) \to H_1(T)$ is injective. $H_1(X) \cong \mathbb{Z}^2 = \mathbb{Z}\{o\} \oplus \mathbb{Z}\{o\}$. We have a map from $T \to o$ and o by projection. So we can construct a map from $H_1(T) \to H_1(o) \oplus H_1(o)$ by $\oplus$ the induced maps of the projections. Since we can map from $H_1(X) \to H_1(T) \to H_1(o) \oplus H_1(o)$ where we map from each generator to itself, then we see that the first map must be an injection.

Therefore, we see that $H_1(X) \to H_1(T)$ is an isomorphism, so the map $H_2(T,X) \to H_1(X)$ is 0., and our homology follows above.

Remark. *We note that there is no map from $T \to X$ since we would have to puncture a hole. But, we can map $H_1(T) \to H_1(X)$ by mapping on the generators of the red and green circles.*

2.3 Singular Homology

Homology is a composition of two functors $\underline{Top} \xrightarrow{C_*} \underline{ChainCplx} \xrightarrow{H_*} \underline{abgp}$, where the categories are topology, chain complexes, and abelian groups. We often say H_* from $\underline{Top}$ to $\underline{abgp}$.

How do we define C_*? Define Δ^n to be an n-simplex. Let $C_n(X) := \mathbb{Z}\{\hom(\Delta^n, X)\}$, which is the free abelian group over all continuous functions $\Delta^n \to X$. These are $a_1[f_1] + a_2[f_2] + \cdots + a_n[f_n]$ where we have formal variables.

Remark. *This is Homology with $\mathbb{Z}$ coefficients. We can do this with any other G. These are normally $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}/p$.*

We need to define $d_n : C_n \to C_{n-1}$ on the generators. This is by $d_n([f_n]) = \sum_{i=0}^{n}(-1)^i[f|_{\{t_i=0\}} : \Delta^{n-1} \to X]$. Note that $d_n \circ d_{n+1} = 0$. We define $H_*(X) := H_*(C_*(X))$.

Ex. *We calculate the homology of a point. Note that* $\hom(\Delta^n, pt) = \{Constant\ Map\}$. *We see that $C_n(pt) = G\{Constant\ Map\} = G$ by picking a generator. Note that $d_n([const]) = \sum_{i=0}^{n}(-1)^i[const] = \sum_{i=0}^{n}(-1)^i$ and this is 0 if n is odd and 1 if n even. Note that 1 occurs when $G \cong \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/n$.*

The Chain complex is $\ldots \xrightarrow{0} G \xrightarrow{1} G \xrightarrow{0} G \to 0 \to \ldots$. Where we go to 0 when we move to $n = -1$. For n odd, we see that $G \xrightarrow{1} G \xrightarrow{0}$ and since $\ker = G, \mathrm{im} = G$ *then $H_n = 0$. For n even and > 0. then* $\ker = 0$ *so $H_n = 0$. At $n = 0$, the sequence is $G \xrightarrow{0} G \to 0$ and since* $\ker = G$ *and* $\mathrm{im} = 0$ *we have $H_0 \cong G$.*

We claim that this is a Homology theory. We need to create this for pairs and boundary maps. Since C_* is functorially, we just need to find a functor from $\underline{Pairs} \to \underline{Ch}$

Def. *A map of chain complex $f_* : C_* \to D_*$ is a map $f_n : C_n \to D_n$ s.t. the following commutes*

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "d_n"] & D_n \arrow[d, "d_n"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1} \end{tikzcd}$$

Remark. *Given a map $f : X \to Y$ we get a map $C_*(X) \to C_*(Y)$ given by $[\Delta^n \to X] \to [\Delta^n \to X \xrightarrow{f} Y]$. This is compatible with the differential because composition with f and restriction to a subset of the domain commute. This map is injective for all n if f is injective.*

Given (X, A), define $C_n(X, A) := C_n(X)/C_n(A)$ and the d_n is just the differential on $C_*(X)$. Given a map of pairs $(X, A) \to (Y, B)$ we get a map $C_*(X, A) \to C_*(Y, B)$, so we see that C_* is a functor.

Lemma (Snake Lemma)**.** *Given a diagram*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

Then there exists an exact sequence

$$\ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c$$

Recall that $\operatorname{coker} f \cong B/\operatorname{im} f$.

Proof. The things to check are that the maps from the kernels exist, the maps from the cokernels exist, the map from $\ker c \to \operatorname{coker} a$ exists, and that the sequence is exist. We show the green map exists and is exact at $\ker c$.

$z \in \ker c \subseteq C$. Since $B \to C$ is surjective by exactness, there is a $z' \in B$ s.t. $z' \to z$. We see that $b(z') \in B'$ and since $b(z') \to 0$ there is an $a' \in A'$ s.t. $a' \to [a']$ in $\operatorname{coker} a$. We say $z \to [a']$.

Let $z \to 0 = [a']$. Then $a' \in \operatorname{im} a$. So there exists an $\widehat{a} \in A$ s.t. $a(\widehat{a}) = a'$. We know that $\widehat{a} \to \widehat{b}$ and by commutativity $b(\widehat{b}) = b(z')$. So $b(z' - \widehat{b}) = 0$ so $z' \in \widehat{b} \in \ker B$. We also see that the image of $\ker c$ of $z' - \widehat{b}$ is z since $\widehat{b}$ maps to 0 by exactness. Therefore, $z' - \widehat{b} \to c$ so we have $z \in$ im and $\operatorname{im}(\ker b) \supseteq \ker$. To show that the image is contained in the kernel since we see that $a' = 0$. □

Corrollary. *Suppose we are given an exact sequence of chain complexes: A_*, B_*, C_* with $f_i : A_i \to B_i$, $g_i : B_i \to C_i$ that commmute w/ the differentials and $\forall i$ $0 \to A_i \to B_i \to C_i$ is exact. Then there exists a LES*

$$\cdots \to H_i(A_*) \to H_i(B_*) \to H_i(C_*) \to H_{i-1}(A_*) \to \dots$$

Proof. We have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} & & \\
0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0
\end{array}$$

Note that $\ker d_i \to H_i(A_*)$ is a surjection (since the homology is the kernel mod the image) and similarly the $\operatorname{coker} d_i$ contains $H_{i-1}(A_*)$ since homology is the kernel mod the image. The snake map produces a new map there. □

Def. *(X, A) is a pair of spaces. We have $H_i(X, A) = H_i(C_*(X, A))$ and $C_i(X, A) = C_i(X)/C_i(A)$ with G coefficients of map $\Delta^i \to X$. The snake lemma says that there exists a map $H_i(X, A) \to H_{i-1}(A)$.*

$C_i(X, A)$ is all free abelian groups generated by map $\Delta^i \to X$ whose image is not contained in A. $H_i(X, A) \to H_{i-1}(A)$ is described as elements as follows. An element of $H_i(X, A)$ is represented by an element of $C_i(X, A)$ whose boundary lies entirely in A. The map takes a function to boundary.

This implies that given a map $(X, A) \xrightarrow{f} (Y, B)$ the follow diagram commutes

$$\begin{array}{ccc}
H_i(X, A) & \xrightarrow{f} & H_i(Y, B) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
H_{i-1}(A) & \xrightarrow{(f|_A)_{i-1}} & H_{i-1}(B)
\end{array}$$

This is axiom 3 and we can easily see axiom 4 holds. Axiom 6 is done with barycentric subdivision. In particular, maps from simplices to the space can be decomposed into a sum of maps of small simplices, and we can separate the small simplices from the space X and the space A.

We finalize by proving axiom 5, which is homotopy equivalence of homology.

Def. *Let $f_*, g_* : C_* \to D_*$ be maps between chain complexes. A* ***chain homotopy*** *is a map $h_n : C_n \to D_{n+1}$ s.t. $f_n - g_n = \partial \circ h_n + h_{n-1} \circ \partial$.*

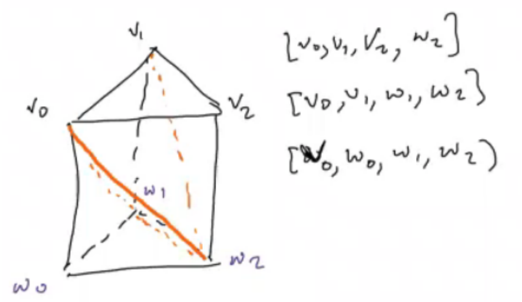
Lemma. *If f_*, g_* are chain homotopic, then the induced maps on the homotopies $H_n C_* \to H_n D_*$ are equal.*

Proof. Let $[\alpha] \in H_n C_*$. This means $\alpha \in C_n$ and $\partial\alpha = 0$. $g_n(\alpha) - f_n(\alpha) = \partial \circ h_n(\alpha) + h_{n-1}\partial(\alpha) = \partial h_n(\alpha) \in \operatorname{im}\partial$. Therefore, since $H_n D_*$ is $\ker / \operatorname{im}$ this is 0 in $H_n D_*$.

In order to show that homotopic maps of pairs induce the same map on homology, it suffices to show that a homotopy of maps of spaces produces a chain homotopy.

Suppose $f, g : X \to Y$ are maps. They are homotopic via a homotopy $H : X \times I \to Y$. $C_n(X) = \mathbb{Z}\{\Delta^n \to X\} \xrightarrow{h_n} C_{n+1}(Y) = \mathbb{Z}\{\Delta^{n+1} \to Y\}$. Given $\phi : \Delta^n \to X$, we get a map $\phi \times I : \Delta^n \times I \to X \times I \to Y$. We want this to be $h_n([\phi])$ but the problem is that this is not a simplex.

We need a triangulation of $\Delta^n \times I$. Think a higher dimensional analogue of this.



In particular let the vertices be given by $v_0, \ldots, v_n$ at $t = 0$ and $w_0, \ldots, w_n$ at $t = 1$. Let the simplices be $[v_0, \ldots, v_i, w_u, \ldots, w_n]$ for $i = 0, \ldots, n$. Let $h_n([\phi]) = \sum_{i=0}^{n} (-1)^i [H(\phi \times I)|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}]$.

Why is this a chain homotopy? We calculate $f_n - g_n = \partial \circ h_n + h_{n-1} \circ \partial$.

$$
\begin{aligned}
(\partial \circ h_n)[\phi] &= \partial \sum_{i=0}^{n} (-1)^i [H \circ (\phi \times I)|_{[v_0,\dots,v_i,w_i,\dots,w_n]}] \\
&= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n+1} (-1)^j [H \circ (\phi \times I)|_{[v_0,\dots,v_i,w_i,\dots,w_n]}] \\
&= \sum_{i=0}^{n} (-1)^i (\sum_{j=0}^{i} (-1)^j [H \circ (\phi \times I)|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i\dots,w_n]}] \\
&+ \sum_{j=1}^{n} (-1)^{j-1} [H \circ (\phi \times I)]|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]})
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
(h_{n-1} \circ \partial)[\alpha] &= h_n \sum_{j=1}^{n} (-1)^j [\phi|_{[v_0,\dots,\widehat{v_j},\dots,v_n]} \\
&= \sum_{j=0}^{n} (-1)^j \sum_{i=0}^{n} (-1)^i [H \circ (\phi|_{[v_0,\dots,\widehat{v_j},\dots,v_n]} \times I)|_{[\dots]}]
\end{aligned}
$$

We see ther terms with a duplicated index cancel out between sums. Terms without duplicated index cancel out in top sum except when term is all v or w, which will be $f_n - g_n$. □

Here are two applications of homology.

Theorem 2.4. *If nonempty open sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are homeomorphic then $m = n$.*

Proof. Let $x \in U$. By excision, $H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$. By the LES of a pair $H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m - \{x\}) \cong \widetilde{H}_{k-1}(S^{m-1})$ since $\mathbb{R}^m$ has mostly trivial homology groups and $\mathbb{R}^m - \{x\} \simeq S^{m-1}$. This implies that $U \setminus \{x\} \cong V - \{f(x)\}$ but these are two spheres (in homology). In particular we need $m - 1$ to equal $n - 1$ so $m = n$. □

Theorem 2.5 ((Brower Fixed Pt))**.** *Let $f : D^n \to D^n$ be an arbitrary map. Then f has a fixed point.*

Proof. Suppose it does not. Define $r : D^n \to S^{n-1}$ by projecting the ray from $f(x) \to x$ onto the sphere. This is a retraction from $D^n \to S^{n-1}$. But such a retraction can't exist as the map $H_k(A) \to H_k(X)$ is an injection for all k if $A \hookrightarrow X \xrightarrow{\exists r} A$. $H_{n-1}(S^{n-1}) \cong Z$ and $H_{n-1}(D^n)$. There are no injections from $\mathbb{Z} \to 0$. $\square$

2.4 Cellular Homology

Recall that in the homework we have that if $f, g : A \to B$ are homotopic then $X \cup_f B \simeq X \cup_g B$.

Def. *A space X is obtained from A by* ***attaching an n-cell*** *if $\exists$ a map $f : S^{n-1} \to A$ s.t. $X \cong A \cup_f D^n$. f is the* ***attaching map*** *of D^n. The image in X of $(D^n)^\circ$ is called a* ***cell****.*

Ex. *Consider the following, where we add a disk to a circle.*

Def. *A* ***CW-structure*** *on a space X is a union of spaces $\emptyset = X^{-1} \subseteq X^0 \cdots \subseteq X^n$ s.t. $\forall n$ X^n is obtained from X^{n-1} by attaching n-cells, $X = \bigcup X^n$. X has the weak topology w.r.t. this union, in particular $C \subseteq X$ is closed if $C \cap X^n$ is closed for all n.*

X is a ***CW-complex*** *if there exists a CW-structure on X.*

A CW-complex is ***n-dimensional*** *if $X^N = X$ for all $N \geq n$. It is* ***finite*** *if there are finitely many cells.*

A ***CW-pair*** *is a pair (X, A) where X is a CW-complex and A is a subcomplex.*

We now show why these CW complexes are good for homology.

Let X be a finite n-dimensional CW complex, and let A be a subcomplex containing all but a single n-cell. We see that $H_k(X, A) \cong \widetilde{H}_k(X/A) \cong \widetilde{H}_k(S^n)$ and this is known. This has a LES

$$\to H_{k+1}(X,A) \to H_k(A) \to H_k(X) \to H_k(X,A) \to \dots$$

If $k, k+1 \neq n$ then both endpoints are 0 and $H_k(X) \cong H_k(A)$. Adding an n-cell can only affect homology at H_n and H_{n-1}.

Up to homotopy equivalence, the spaces you can produce by attaching n-cell for X are classified by homotopy classes of maps $S^{n-1} \to X$. Here we see that

$$H_{n+1}(X,A) \to H_n(A) \hookrightarrow H_n(X) \xrightarrow{?} H_n(X,A) \cong \mathbb{Z} \xrightarrow{?} H_{n-1}(A)$$

we need to find the question marks.

Ex. *S^n is 1 0-cell and 1 n-cell attached by the constant map $S^{n-1} \to pt$. S^2 is a point and everything else is a n-cell (can see by stereographic projection).*

Note that $H_0(S^n) \cong H_n(S^n)$ and we have 1 0-cell and 1 n-cell.

Ex. *Recall torus (square top glued to bottom and left glued to right). X^0 is the pt, X^1 is a figure 8 of the sides of the rectangle, and X^2 is a torus by adding a ball which corresponds to the center of the rectangle. Again note the homology matches the number of cells.*

Ex. *Another CW-complex on S^1 is three points with arcs connecting them. This is useful for constructing pairs of spaces.*

Ex. *$\mathbb{CP}^n$ is lines in $\mathbb{C}^{n+1}$. A way of thinking about this is "half" of the sphere.*

Note that $\mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$ (think when $z_n = 0$). Claim that $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is an open ball of dimension $2n$.

For any point $[z_0 : \cdots : z_n] \in \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$. Choose a representative $(x_0, \dots, x_n)$ of norm 1 s.t. x_n is real and nonnegative. $f : D^{2n} \to \mathbb{CP}^n$ by mapping $(y_1, \dots, y_{2n}) \to [y_1 + iy_2 : \cdots : y_{2n-1} + iy_{2n} : \sqrt{1 - |y|^2}]$. For $|y| < 1$ this is injective. When $x_n \neq 0$ this has continuous inverse. We see that $f_{S^{2n-1}} : S^{2n-1} \to \mathbb{CP}^{n-1}$ maps to last coordinate to 0, so this is attachment. By induction $\mathbb{CP}^n$ has a cell structure with 1 cell in each of the even dimensions.

Consider the pair (X^n, X^{n-1}). We have a LES

$$H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \to H_{n-1}(X^n, X^{n-1})$$

We see that the first is 0 (note that $n > n-1$ and cells only change homology up to their location). The last is also 0 since $H_{n-1}(X^n, X^{n-1}) \cong \widetilde{H}_{n-1}(X^n/X^{n-1})$. Note that $X^n/X^{n-1} \cong \bigsqcup_{ncells} D^n / \bigsqcup_{ncells} S^{n-1} \cong \bigvee_{ncells} S^n$ since everything else besides the circles trivially contracts and we are left with the disks. Since homology of wedge is $\oplus$, this is 0 since it is $n-1$ homology. Similarly, $H_n(X^n, X^{n-1}) \cong \oplus_{ncells}\mathbb{Z}$.

We define a chain complex $C_*^{CW}(X)$ by $C_n^{CW}(X) := H_n(X^n, X^{n-1})$. We can construct the sequence by gluing together exact sequences like follows

$$\begin{tikzcd}
 & & & 0 & & \\
0 \arrow[dr] & & H_{n-1}(X^n) \arrow[ur] & & & \\
 & H_n(X^n) \arrow[ur, two heads] \arrow[dr] & & & & H_{n-2}(X^{n-2}) \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur] \arrow[rr, "\partial"] & & H_n(X^n, X^{n-1}) \arrow[rr, "\partial"] \arrow[dr] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur] & \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, hook] \arrow[dr] & & \\
 & & & & H_{n-1}(X) \arrow[dr] & \\
 & & & & & 0
\end{tikzcd}$$

and taking ∂ to be the resulting maps since these are C_n^{CW}.

What is the homology of these chain complexes? We see by the SES that the kernel of the second ∂ is $H_n(X^n)$ (since the map from $H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$) is injective. Similarly, since the map $H_n(X^n) \to H_n(X^n, X^{n-1})$ is injective, we find that the image is the image of the map from $H_{n+1}(X^{n+1}) \to H_n(X^n)$. We see that the homology is $H_n(X^n)/\ker(H_n(X^n) \to H_n(X^{n+1})) \cong H_n(X^{n+1})$ and we note that $H_n(C_*^{CW}(X)) \cong H_n(X^{n+1}) \cong H_n(X)$.

Ex. $H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2, \ldots, 2n \\ 0 & otherwise \end{cases}.$

The above is terrible though and not great for computing anything. We now try to show how to compute stuff better.

$C_n^{CW}(X) = \mathbb{Z}^r$ where r is the number of n-cells. Note that $C_{n-1}^{CW} \cong \mathbb{Z}^{r'}$ so to compute $\partial : \mathbb{Z}^r \to \mathbb{Z}^{r'}$. To compute it we just need to know what happens to each generator.

Let α be an n-cell and β be an $n-1$ cell. α is attached by $f : S^{n-1} \to X^{n-1}$. The induced map $\widetilde{f} : S^{n-1} \xrightarrow{f} X^{n-1} \to S_\beta^{n-1}$. The second map collapses X^{n-2} and all non-β cells to a point. This is a map from a $n-1$ sphere to an $n-1$ sphere.

Def. *Let $f : S^{n-1} \to S^{n-1}$ be map. f induces $f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$. The **degree** of f is the integer that this multiplies by.*

Consider the following diagram: (Hatcher p 141 top, Weintraub proof of 4.2.11).

$$\begin{tikzcd}
H_n(D^n, S^{n-1}) \arrow[r, "\partial"] \arrow[d, "F_*"] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[rr, "\widetilde{f}_*"] \arrow[d, "f_*"] & & \widetilde{H}_{n-1}(S^{n-1}) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "\partial^{CW}"] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[rr] \arrow[d] & & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "(Q_B)_*"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[rr, "\cong"] & & H_{n-1}(X^{n-1}/X^{n-2}, *)
\end{tikzcd}$$

For a map $f : S^{n-1} \to X^{n-1}$ we induce a map $F : (D^n, S^{n-1}) \to (X^n, X^{n-1})$ where $S^{n-1} \to X^{n-1}$ by f and $D^n \to X^n$ maps the D^n to α.

The rest of the CD is defined as follows. The upper-left hand square is defined by axiom 2, $\widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1})$ is by definition $\widetilde{f}$, Q_β is the collapse to β we defined above, $\widetilde{H}_{n-1}(X^{n-1}) \to \widetilde{H}_{n-1}(X^{n-1}/X^{n-2})$ by the quotient, and $H_{n-1}(X^{n-1}, X^{n-2}) \to H_{n-1}(X^{n-1}/X^{n-2}, *)$ because it is a good pair. The map on the generators is the top row, and we want to find ∂^{CW}.

We get that $\partial[\alpha] = \sum_\beta (\deg Q_\beta \circ f)[\beta]$. The degree is the multiplication constant of the induced map $f_* : \mathbb{Z} \to \mathbb{Z}$.

In general, computing degree is hard. We first present some basic properties of the degrees

1. $\deg 1 = 1$

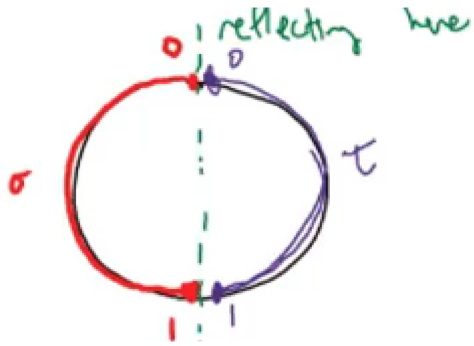
2. $\deg f = 0$ if f is not surjective.

   If $f : S^{n-1} \to S^{n-1}$, then we see that f actually maps through $S^{n-1} \setminus \{x\}$. This means that the induced map f_* maps through $H_{n-1}(S^{n-1} \setminus \{x\}) = 0$.

3. $f \sim g$ iff $\deg f = \deg g$. The reverse direction is pretty hard to show.

4. $\deg(g \circ f) = (\deg g) \circ (\deg f)$.

5. $\deg f = -1$ is f is a reflection. Since $f^2 = 1, (\deg f)^2 = 1$ and so $\deg f = \pm 1$. To see that this is -1, take a singular-homology generator which is symmetric w.r.t. the reflection. Consider the 1 dimensional case

   H_1 is $[\sigma] - [\tau]$, and reflections swap $[\tau] - [\sigma]$ (which shows that it is -1).

   In general, take the quadrants of the sphere (the above case would have 4). By ordering the axes to orient, we see that reflections across the x axis will swap orientation.

6. The antipodal map has degree $(-1)^n$. This is because it is the composition of n reflections.

7. If f has no fixed points, then $\deg f = (-1)^n$. This is because $f \sim$ antipodal. Draw $f(x) \to -x$ and project, and this is the homotopy.

The following is an application of degree maps.

Theorem 2.6 (Hairy-Ball Theorem)**.** *S^n has a continuous vector field iff n is odd.*

Proof. If n is odd then $(x, y) \to (-y, x)$ for each pair of coordinates. This works.

If n is even, if a continuous vector field exists, then it is homotopic to the antipodal map. Take a vector field and follow the direction to the opposite. But, this is impossible since the degree of the antipode is $-1 \neq 1$. □

Corrollary. *$\mathbb{Z}/2$ is the only group that can act freely with no fixed points on S^n if n is even.*

Proof. Take a map $\deg : G \to \mathbb{Z}/2$. Since everything in G is invertible, every element can only have degree ± 1 (since the only invertible maps $\mathbb{Z} \to \mathbb{Z}$ are ± 1). This maps all non-identity maps to -1. Therefore, there is only one non-identity element by the above. □

Now, we show how to compute the degree map. We use local degrees.

Let $f : S^{n-1} \to S^{n-1}$ and y be in the image. Let y have a finite preimage $x_1, \dots, x_m$, and elt $U_1, \dots, U_m$ be disjoint neighborhoods of x_i where $F(U_i - x_i) \subseteq V - y$ for all i. This induces the following

$$\begin{tikzcd}
 & H_{n-1}(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[d] \arrow[dl] & H_{n-1}(V, V - y) \arrow[d] \\
H_{n-1}(S^{n-1}, S^{n-1} - x_i) & H_{n-1}(S^{n-1}, S^{n-1} - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_{n-1}(S^{n-1}, S^{n-1} - y) \\
 & H_{n-1}(S^{n-1}) \arrow[r, "f_*"] \arrow[u] \arrow[ul] & H_{n-1}(S^{n-1}) \arrow[u]
\end{tikzcd}$$

We can see that many maps are isomorphic. The top-right is isomorphic by excision, bottom right is by LES of $(S^{n-1}, S^{n-1} - y)$, top and bottom left also follow similarly (excision and LES of $(S^{n-1}, S^{n-1} - x_i)$). The only questions are everything else.

The top is defined as the local degree. It is always ± 1, as can be checked, since f is locally invertible.

The top center arrow is not an isomorphism. This is because, as can be checked, all groups are $\mathbb{Z}$ except the center group, which is $\mathbb{Z}^m$. Therefore, we see that the maps are followed. The center top map is the inclusion since $H_{n-1}(S^{n-1}, S^{n-1} - f^{-1}(y)) \cong \bigoplus H_{n-1}(U_i, U_i - x)$. The left map p_i is projection onto the i-th coordinate, and the bottom center map it the diagonal map $1 \to (1, \ldots, 1)$.

The bottom map f_* can then to be seen as follows. If we sum over the values of i in the top row, we see that the inclusion becomes a isomorphism and $\deg f = \sum$ local degrees.

Ex. $\mathbb{RP}^n \cong S^{n+1}/\{\pm 1\}$.

A CW structure on S^{n+1} is as follows. We start with 2 0-cells, 2 1-cells, etc.. to 2 n-cells where these are thought of as hemispheres. The ± 1 action glues each cell together.

Therefore, we see that $\mathbb{RP}^n$ has a CW structure with 1 cell in each of the dimensions. Therefore, we see that $C_*^{CW} = 0 \to \mathbb{Z}(n) \to \cdots \to \mathbb{Z}(0) \to 0$. We need to compute the degrees of the attaching maps. Note that $\mathbb{RP}^n$ is a single n-cell attached to $\mathbb{RP}^{n-1}$.

∂D^n hits the $n-1$ cell twice. To imagine this, note that when adding a hemisphere, we attach on the circle. But, if both sides are related, we must instead hit the structure twice. On one side, the local degree is 1 (think of this as the identity), and the other side is the local degree is the degree of the antipodal map $(-1)^n$. Therefore, the degree of the attaching map $1 + (-1)^n$, and in particular $C_i^{CW}(\mathbb{RP}^n) \to C_{i-1}^{CW}(\mathbb{RP}^n)$ is $1 + (-1)^i$. Therefore, we see that the maps are

$$0 \to \mathbb{Z}(n) \xrightarrow{1+(-1)^n} \mathbb{Z} \xrightarrow{1+(-1)^{n-1}} \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}(0) \to 0$$

Therefore, $H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i \text{ odd} < n \\ 0 & i \text{ even} < n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}$. *$H_i(\mathbb{RP}^\infty)$ is the above with no boundary conditions.*

Remark. *All the presented homologies are the same (singular, simplicial, cellular). We proved this in the hw.*

2.5 Homology with Different Coefficients

We have just computed $\mathbb{RP}^n$. Now let's see what happens when $H_0(*) = \mathbb{Z}/2$. Claim: $C_*^{sing}(X;\mathbb{Z}/2) := C_*^{sing}(X) \otimes \mathbb{Z}/2$. Can define the same for CW homology. Then our chain would be $\mathbb{Z}/2 \to \mathbb{Z}/2 \cdots \to \mathbb{Z}/2 \to 0$ and all of our maps would be 0, so $H_*(\mathbb{RP}^n;\mathbb{Z}/2) = \mathbb{Z}/2$, and we see that $H_i(\mathbb{RP}^n;\mathbb{Z}/2) \neq H_i(\mathbb{RP}^n) \otimes \mathbb{Z}/2$. This leads us naturally into two questions:

Q: How do we compute $H_*(X;\mathbb{Z}/2)$ from $H_*(X)$?

Q: Can we compute $H_*(X \times Y)$ from $H_*(X)$ and $H_*(Y)$?

Def. *A vector space* ***tensor product*** $V \otimes W$ *is space with basis* $v_i \otimes w_j$.

Theorem 2.7. $\hom^{lin}(V \otimes W, Z) \cong \hom^{bilin}(V \times W, Z)$.

Def. *A, B abelian groups then $A \otimes B$ is* $\hom(A \otimes B, C) \cong \hom^{bilin}(A \times B, C)$, *where bilinearity means that $\varphi(a,b)$ is a homomorphism $B \to C$ for any fixed a and $A \to C$ for any fixed b.*

$A \otimes B = \left(\sum_{i \in I : I \text{ finite}} a_i \otimes b_i\right) / \sim$, where $\sim$ is bilinearity ie $(a + a') \otimes b = a \otimes b + a' \otimes b$.

Ex.

- *$\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ with $1 \otimes 1 \to 1$.*
- *$\mathbb{Z} \otimes A \cong A$ with $1 \otimes a \to a$.*
- *$\mathbb{Z}/p \otimes \mathbb{Z}/q = 0$ if $p \neq q$ and $\mathbb{Z}/p$ if $p = q$.*
- *$\mathbb{Z}/p \otimes \mathbb{Q} \cong 0$.*
- *$\mathbb{Z}[x] \otimes \mathbb{Z}[y] \cong \mathbb{Z}[x,y]$.*

We see that tensor products are pretty nontrivial. Inna's favorite example is $\mathbb{R} \otimes \mathbb{R}/\mathbb{Z}$, where it's hard to tell if an element is nonzero.

This relates to homology since the homology of $X \times Y$ would be more difficult. One may think that $H_n(X \times Y) \cong H_n(X) \otimes H_n(Y)$, but this is false. Alternatively, $H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$, and this is true with rational coefficients (and is therefore much closer).

Now examine tensor products of chain complexes. Consider C_*, D_*. We let $(C \otimes D)_* : (C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$. What is the boundary? The boundary is given by $d(x \otimes y) = dx \otimes y + (-1)^{|x|} \otimes dy$. $|x|$ is the degree of x and is i if $x \in C_i$. What is the homology?

Prop 2.2. *Suppose C_i is free abelian. For all n, there exists a SES*

$$0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \to H_n(C \otimes D) \to \bigoplus_{i+j=n} \mathrm{Tor}(H_i(C_*), H_{j-1}(D_*)) \hookrightarrow 0$$

and the sequence splits (Hatcher 3B, Weintraub 5.4).

Proof. Consider a special case. First, assume all boundary maps in C_* are 0. Then $C \otimes D \cong (\bigoplus_i C_i) \otimes D$ and the bondary map is just the boundary map on D. Then $H_n(C \otimes D) \cong \bigoplus_i C_i \otimes H_{n-i}(D) \cong \bigoplus_i H_i(C) \otimes H_{n-i}(D)$. This means that if the boundary map is trivial, then there is no Tor term.

Define $Z_i \ker d_i : C_i \to C_{i-1}$, $B_i = \operatorname{im} d_{i+1} : C_{i+1} \to C_i$, Z_* the chain complex of Z_i with 0 boundary map and B_* similarly. We have a SES

$$0 \to Z \to C \xrightarrow{d} B[-1] \to 0$$
$$0 \to Z_i \to C_i \xrightarrow{d_i} B_{i-1} \to 0$$

and this sequence splits. The claim is that if we tensor this with D_* this will still be exact and split.

$$0 \to Z \otimes D \to C \otimes D \to B[-1] \otimes D \to 0$$

is a split SES of chain complexes. $H_n(Z \otimes D) \cong \bigoplus_{i+j=n} H_i(Z) \oplus H_j(D)$ and $H_n(B[-1] \otimes D) \cong \bigoplus_{i+j=n} H_i(B[-1]) \oplus H_j(D) \cong \bigoplus_{i+j=n} H_{i-1}(B) \otimes H_j(D)$. These both follow from the special case above.

Have the following LES

$$\ldots \xrightarrow{i_n} \bigoplus_i Z_i \otimes H_{n-i}(D) \to H_n(C \otimes D) \to \bigoplus_i B_i \otimes H_{n-i-1}(D) \xrightarrow{i_{n-1}} \bigoplus_i Z_i \otimes H_{n-i-1}(D)$$

Have a SES

$$0 \to \operatorname{coker} i_n \to H_n(C \otimes D) \to \ker i_{n-1} \to 0$$

so $\operatorname{coker} i_n \cong \bigoplus_i Z_i \otimes H_{n-i}(D)/\operatorname{im} i_n \cong \bigoplus_i H_i(C) \otimes H_{n-i}(D)$ through snake lemma. So this gives us the first two terms in the proposition.

What is $\ker i_{n-1}$? One possibility is $\ker i_{n-1} = \bigoplus \ker(B_i \otimes H_{n-i-1}(D) \to Z_i \otimes H_{n-i-1}(D))$, which is $\operatorname{Tor}(H_i(C), U_{n-i-1}(D))$. $\square$

Def. *A, B free abelian groups. A **free resolution** of A is a SES $0 \to F' \to F \to A \to 0$ where F', F are free abelian.* $\operatorname{Tor}(A, B) := \ker(F' \otimes B \to F \otimes B)$.

Remark. *Note that $B_i \to Z_i \to H_i(C)$ is a free resolution of $H_i(C)$.*

Lemma. *Tor is independent of the choice of free resolution.*

If A is finitely generated, a free resolution always exists. In general, we work with R-modules, which are much more complicated (think of $\mathbb{Z}$-modules as abelian groups).

$\otimes B$ is a right exact functor: for a free resolution $F' \otimes B \to F \otimes B \to A \otimes B \to 0$ is exact and the homology at $F' \otimes B$ is $\operatorname{Tor}(A, B)$ while the homology every else is 0.

Ex.

- *If A is free then* $\operatorname{Tor}(A, B) = 0$. *We can set $F = A, F' = 0$ and the kernel is* 0.
- $\operatorname{Tor}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$. *More generally,* $\operatorname{Tor}(\mathbb{Z}/\ell, B) \cong \ker(B \xrightarrow{\cdot \ell} B)$ *since we can set $F = \mathbb{Z}, F' = \mathbb{Z}'$ with the map $F' \xrightarrow{\cdot \ell} F$. This also gives us that, for prime p, q,* $\operatorname{Tor}(\mathbb{Z}/p\mathbb{Z}/q) = \mathbb{Z}/p$ *iff $p = q$ (otherwise* 0*).*
- $\operatorname{Tor}(A \otimes A', B) \cong \operatorname{Tor}(A, B) \otimes \operatorname{Tor}(A', B)$.
- $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(B, A)$.

Recall that we proved earlier that there is a LES

$$\xrightarrow{0} \bigoplus_{i+j=n} H_i(C) \otimes H_i(D) \to H_n(C \otimes D) \to \bigoplus \operatorname{Tor} \xrightarrow{0} \bigoplus_{i+j=n-1} H_i(C) \otimes H_i(D)$$

Prop 2.3. *Let X, Y be spaces. There is a map*

$$C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

which induces an isomorphism on homology. This works for any chain complex.

Proof. We're going to prove this using singular homology.

Let $\sigma : \Delta^n \to X$, $\tau : \Delta^m \to Y$ be singular simplices in X, Y respectively with vertices $v_0 \ldots, v_n$, $w_0, \ldots, w_m$. Let P be the set of paths from $(0,0)$ to (n,m) in the xy plane that go up or right. There are $\binom{n+m}{n}$ of these. We will decompose $\Delta^n \times \Delta^m$ into simplices using P.

Construct $\Delta^{n+m} \xrightarrow{p_w} \Delta^n \times \Delta^m$ for $w \in P$. The i-th vertex is mapped to v_{j_i}, w_{k_i}, where $w = (0,0), (j_1, k_1) \ldots, (j_{m+n}, k_{m+n})$.

Define the map $C_*(X) \times C_*(Y) \to C_*(X \otimes Y)$ by mapping $\sigma \times \tau \to \sum_{w \in P} (-1)^{\alpha(w)} (\sigma \times \tau) \circ p_w$. α is the sign of the permutation between all right then all up and the path. This works, and the point of the sign is to accommodate for the $d(\sigma \times \tau)$. ∎

Ex. *$T^1 \cong S^1 \times S^1$. Note that $H_*(S^1)$ is free for all $*$, so* Tor *will be* 0. *Therefore $H_n(T^1) \cong \bigoplus_{i+j=n} H_i(S^1) \otimes H_j(S^1)$ and is easily found to be $\mathbb{Z} : n = 0, 2$ and $\mathbb{Z}^2 : n = 1$.*

More generally, we can compute the homology of the n-torus. In particular $H_m(T^n) \cong \bigoplus_{i_1 + \cdots + i_n = m} H_{i_1}(S^1) \otimes \cdots \otimes H_{i_n}(S^1) = \mathbb{Z}^{\binom{n}{m}}$.

Corrollary (Universal Coefficient Theorem)**.** *Examine $H_n(X; G) = H_n(C_*(X) \otimes G)$ where G is abelian. G in the RHS is the chain complex of G in degree* 0 *and* 0 *otherwise. Then $H_n(X; G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(X), G)$. The G in the product is H_0 of the chain complex concentrated at* 0 *of G, so this is a special case of the formula.*

Ex. *If $H_*(X)$ is free in every degree, then $H_*(X; G) \cong H_*(X)$*

- *$H_i(\mathbb{RP}^n; \mathbb{Z}/2) \cong (H_i(\mathbb{RP}^n) \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}(H_{i-1}(\mathbb{RP}^n), \mathbb{Z}/2)$. We can compute by tensoring $H_i(\mathbb{RP}^n)$. In particular note that if it is $\mathbb{Z}/2$ then* $\mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$ *and if it is $\mathbb{Z}$ then* Tor *is* 0.

3 Cohomology

3.1 Axioms of Cohomology

We noticed that $H_*(S^2 \vee S^4) \cong H_*(\mathbb{CP}^2)$ but these spaces are not homotopic. We want to construct a new topological invariant that is better.

Def. *A **Cohomology Theory** is a sequence of functors $Pairs^{op} \to AbGp$ together with natural transformations $\delta_i : H^i(A, \emptyset) \to H^{i+1}(X, A)$ for pairs (X, A). For a map $f : (X, A) \to (Y, B)$ we write $f^i : H^i(Y, B) \to H^i(X, A)$ as the induced map. Note that this is contravariant.*

The axioms are given by

- *Axiom 3: Naturality of δ. For $f : (X, A) \to (Y, B)$ we have the cd*

$$\begin{array}{ccc} H^i(B) & \xrightarrow{\delta_i} & H^{i+1}(Y, B) \\ \big\downarrow{\scriptstyle f^i} & & \big\downarrow{\scriptstyle f^{i+1}} \\ H^i(A) & \xrightarrow{\delta_i} & H^{i+1}(X, A) \end{array}$$

- *Axiom 4: LES $\ldots \leftarrow H^i(A) \leftarrow H^i(X) \leftarrow H^i(X, A) \xleftarrow{\delta_{i-1}} H^{i-1}(A) \leftarrow \ldots.$*
- *Axiom 5: If f, g homotopic, then $f^i = g^i$.*
- *Axiom 6 (excision): If $U \subseteq A$ and $\overline{U} \subseteq A^\circ$ and $f : (X - U, A - U) \to (X, A)$ is the inclusion then $f^i : H^i(X, A) \to H^i(X - U, A - U)$ is an isomorphism.*
- *Axiom 7: $H^i(pt) = 0$ for all $i \neq 0$.*

Theorem 3.1. *$H^i(X, A) \cong \widetilde{H}^i(X/A)$ if A has a nbhd that deformation retracts to A.*

Def. $\widetilde{H}^i(X) = \operatorname{coker}(H^i(pt) \to H^i(X))$

Ex. *We will calculate the cohomology of spheres. Consider the pair (CX, X). We have a LES*

$$H^{i-1}(CX) \to H^{i-1}(X) \to H^i(CX, X) \to H^i(CX)$$

CX is contractible and we proved. So $i \neq 0$ implies that $H^i(CX) = H^i(pt) = 0$. For $i > 1$, $H^i(CX, X) \cong H^{i-1}(X)$. Using the definition of cokernel, we get that $H^i(CX, X) \cong \widetilde{H}^{i-1}(X)$ (???).

We note that $\widetilde{H}^i(S^0) \cong H^0(pt)$ and this implies that $\widetilde{H}^i(S^n) = H^0(pt)$ if $i = n$ otherwise 0. This implies that $H^i(S^n) = H^0(pt)$ for $i = 0, n$ and 0 otherwise since $CX/X \cong \Sigma X$.

This looks very wrong to me.

Remark. *There are some issues.*

We showed that if $X = \bigsqcup_{i \in I} X_i$ then $H_*(X) \cong \bigoplus_{i \in I} H_*(X_i)$. For cohomology, $H^*(X) \cong \prod_{i \in I} H^*(X^i)$.

Chain complexes in cohomology will not always be free. This is because our stuff is dual, and duals of free groups may not be free.

3.2 Cohomology Complexes

Def. *Take a chain complex for any homology (singular, simplicial, CW, etc...). Define $C^n(X) := \hom(C_n(X), \mathbb{Z})$ to be the **cochain complex**. The **coboundary map** $d^n : C^n(X) \to C^{n+1}(X)$ is $[f : C_n(X) \to \mathbb{Z}] \to [f \circ d : C_{n+1}(x) \xrightarrow{d} C_n(X) \to \mathbb{Z}]$.*

Def. *$H^n(X; \mathbb{Z}) = \ker d^n / \operatorname{im} d^{n-1}$. $H^n(X; G) = H^n(C^*(X; G))$ where $C^n(X : G) = \hom(C_n(X), G)$.*

All proofs of axioms work the same except LES. The reason why is that $C_*(X, A) := \operatorname{coker}(C_*(A) \to C_*(X))$ and $C^*(X, A) := \ker(C^*(X) \to C^*(A))$. But one may ask why we don't just use $(\hom(C_*)(X, A) \to \mathbb{Z})$. The answer is that if you apply hom to an exact sequence $0 \to A \to B \to C \to 0$, then $0 \to \hom(C, \mathbb{Z}) \to \hom(B, \mathbb{Z}) \to \hom(A, \mathbb{Z}) \to 0$ may not be exact. Ie $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. For example, this holds for topological spaces.

Ex. *Consider $\mathbb{RP}^3$. Recall that*

$$C_*^{CW}(\mathbb{RP}^3) = \cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and $H_*(\mathbb{RP}^3)$ *is thus* $\mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}$. *We also have that*

$$C^*_{CW}(\mathbb{RP}^3) = \ldots \leftarrow 0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

The maps are generated by precomposing with d *so the maps take the form above. The cohomologies are thus* $H^*(\mathbb{RP}^3) = \mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$. *Note that* $\mathbb{Z}/2$ *is not* $\hom(H_2(\mathbb{RP}^3), \mathbb{Z})$.

But, in special cases we can use the following:

Lemma. *If* $A \hookrightarrow B \twoheadrightarrow C$ *splits ie there exists* $C \to B$, *then* $0 \to \hom(C, \mathbb{Z}) \to \hom(B, \mathbb{Z}) \to \hom(A, \mathbb{Z})$ *is exact.*

Proof. Need to show that $\hom(B, \mathbb{Z}) \to \hom(A, \mathbb{Z})$ is surjective. Use splitting to show that there exists a map $B \to A$ and so there exists a map $\hom(A, \mathbb{Z}) \to \hom(B, \mathbb{Z})$ that shows it is exact. □

So if the sequence splits, you're fine. In particular, for singular homology $C_*(X) \cong C_*(A) \oplus C_*(X, A)$, so this gives us a good splitting for this to work (this works in general but this is easy to see).

So we have an exact sequence of cochain complexes

$$0 \to C^*(X, A) \to C^*(X) \to C^*(A) \to 0$$

using snake lemma. We finish by showing the universal coefficient theorem.

Theorem 3.2 (Universal Coefficient Thm-Tor Version)**.** *If* X *has finitely many cells and* G *is finitely generated, there is a natural split exact sequence*

$$0 \to H^n(X) \otimes G \to H^n(X; G) \to \mathrm{Tor}(H^{n+1}(X), G) \to 0$$

Proof. If X has finitely many cells and we use $C^{CW}_*(X)$, the dual will still be free. So the theorem applies. If G is finitely generated, we can run the proof again and note all that was actually used was that $C^*(X)$ is torsion free, which it is (using $C^*X; G) = C^*(X) \otimes G)$. □

Ex. *$C_*^{CW} = 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$ and homology is $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$. C^*_{CW} are $\mathbb{Z} \xleftarrow{0} \mathbb{Z}^2 \xleftarrow{0} \mathbb{Z}$. In particular, this case has H^* the same as H_*, and in particular $H^* \cong \hom(H_*, \mathbb{Z})$.*

Ex. *If differentials are 0 in C_*^{CW}, then $\hom(C_*^{CW}, \mathbb{Z}) = C_n^{CW}$ and $H^n(C^*_{CW}) = H^n(\hom(C_*^{CW}, \mathbb{Z})) = \hom(C_n^{CW}, \mathbb{Z}) = \hom(H_n(C_*^{CW}), \mathbb{Z})$. But this isn't always the case.*

Ex. *Take S^2 and attach a 3-cell with map of degree 3. Take S^2 and collapse equator and do it on a component to get $S^2 \to S^2 \vee S^2 \to S^2 \vee S^2 \vee S^2$ and fold together.*

We get that $C_*^{CW} = Z \xrightarrow{3} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}$. $H_* = 0, \mathbb{Z}/3, 0, \mathbb{Z}$. $C^*_{CW} = \mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{0} 0 \xleftarrow{0} \mathbb{Z}$. $H^* = \mathbb{Z}/3, 0, 0, \mathbb{Z}$.

With $\mathbb{Z}/3$ coefficients, we see that C^*_{CW} is $\mathbb{Z}/3 \xleftarrow{0} \mathbb{Z}/3 \xleftarrow{0} 0 \xleftarrow{0} \mathbb{Z}/3$. Therefore, H^* would be $\mathbb{Z}/3, \mathbb{Z}/3, 0, \mathbb{Z}/3$.

With $\mathbb{Z}/2$ coefficients C^*_{CW} is $\mathbb{Z}/2 \xleftarrow{1} \mathbb{Z}/2 \xleftarrow{0} 0 \xleftarrow{0} \mathbb{Z}/2$, so the cohomologies are $0, 0, 0, \mathbb{Z}/2$.

There is a map $e_v : C^n(X) \times C_n(X) \to \mathbb{Z}$. In particular, for an $f \in C^n(X)$ and $\gamma \in C_n(X)$, then we map to $f(\gamma)$. This is called the evaluation map.

Lemma. *e_v induces a map from $H^n(X) \otimes H_n(X) \to \mathbb{Z}$.*

Proof. Weintraub 5.5. We just need to check that e_v is 0 if γ is a boundary and if f is a co-boundary. □

We note that $\otimes A$: abGp $\to$ abGp is left adjoint to $\hom(A, \cdot)$: abGp $\to$ abGp. This implies that ev induces an adjoint map $H^n(X) \xrightarrow{ev'} \hom(H_n(X), \mathbb{Z})$.

Theorem 3.3 (Universal Coefficient Theorem)**.** *For all X, there exists a split exact sequence*

$$0 \to \operatorname{Ext}(H_{n-1}(X), G) \to H^n(X; G) \to \hom(H_n(X), G) \to 0$$

Proof. Let $C_* = C_*(X)$, Z_n the cycles, and B_n the boundaries. There exists a SES

$$0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0$$

Get an exact sequence of chain complexes $0 \to Z_* \to C_* \to B_*[-1] \to 0$. We apply $\hom(\cdot, G)$ to get a SES (since we have a split SES)

$$0 \to \hom(B_*[-1], G) \to \hom(C_*, G) \to \hom(Z_*, G) \to 0$$

Here the coboundary maps are still 0. With this SES of chain cocomplexes, we get a LES

$$\to \hom(Z_{n-1}, G) \xrightarrow{i_n} \hom(B[-1]_n, G) \to H^n(X; G) \to \hom(Z_n, G) \xrightarrow{i_{n+1}} \hom(B[-1]_{n+1}; G) \to$$

And with this we get a SES at $H^n(X; G)$.

$$0 \to \operatorname{coker} i_n \to H^n(X; G) \to \ker i_{n+1} \to 0$$

We analyze the coker to get the ext term and analyze the ker term to get $\hom(H_n(X), G)$. The rest of the proof can be found in Hatcher 3.1. $\square$

Def. *Let A, B be abelian groups. Let $0 \to F' \to F \to A \to 0$ be a SES with F, F' free. Apply $\hom(\cdot, B)$: $0 \to \hom(A, B) \to \hom(F, B) \xrightarrow{i} \hom(F', B)$. Define $\operatorname{Ext}(A, B) : \operatorname{coker} i$.*

Here are some properties of ext:

(1) If A is free, $\operatorname{Ext}(A, B) = 0$.

(2) $\operatorname{Ext}(\bigoplus_\alpha A_\alpha, B) \cong \prod_\alpha \operatorname{Ext}(A_\alpha, B)$ and $\operatorname{Ext}(A, \prod_\beta B_\beta) = \prod_\beta \operatorname{Ext}(A, B_\beta)$.

(3) $\operatorname{Ext}(\mathbb{Z}/p, B) \cong B/pB$.

Remark. Ext *is not symmetric, even though* Tor *is.*

There are some consequences of Universal Coefficient Thm

Theorem 3.4. *X space s.t. $H_n(X) \cong F_n \oplus T_n$ where F_n is free and T_n is torsion. Then $H^n(X) \cong F_n \oplus T_{n-1}$.*

Theorem 3.5. *If $f : X \to Y$ induces an isomorphism on H_n for all n, then it induces an isomorphism $H^n(Y;G) \to H^n(X;G)$ for all n.*

Theorem 3.6 (Kunneth Theorem for Cohomology)**.** *Suppose X, Y have finitely generated homology. Then there is a natural split exact sequence*

$$0 \to \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \to H^n(X \times Y) \to \bigoplus_{i+j=n} \mathrm{Tor}(H^i(X), H^i(Y)) \to 0$$

If we want to work with coefficients on a ring R, then we must work with Tor^R*, where we resolve by free R-modules. In particular, for a field $\mathbb{F}$ then*

$$H^n(X \times Y; \mathbb{F}) \cong \bigoplus_{i+j=n} H^i(X;\mathbb{F}) \otimes H^j(Y;\mathbb{F})$$

3.3 Cohomology Ring

Remark. *You may think to yourself that cohomology isn't more useful than homology since homology groups can determine cohomology groups. However, cohomology has another construct.*

Def. ***cohomology ring*** *$H^*(X;R)$ where R is a ring to be the graded ring w/graded commutative multiplication.*

If $\alpha \in H^n(X)$ let $|\alpha| = n$. Define $H^*(X) \bigoplus_{n\in\mathbb{Z}} H^n(X)$. Let $\alpha \in H^*(X)$ to mean that $\alpha \in H^n(X)$.

Recall that the Eilienberg-Ziller map $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ is constructed by taking a $f : \Delta^n \to X$ and $g : \Delta^m \to Y$ and we decomposed $\Delta^n \times \Delta^m$ into a sum of simplices. Let this map be E.

There exists a map $F : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ called the alexander-whitney map. For a $\sigma : \Delta^n \to X \times Y$, this consists of the $\sigma_X : \Delta^n \to X$ and $\sigma_Y : \Delta^n \to Y$. We map this to the sum

$$\sum_{i=0}^{n} [\sigma_x|_{[v_0,\dots,v_i]}] \otimes [\sigma_Y|_{\{v_i,\dots,v_n\}}]$$

Theorem 3.7 (Eilienberg-Ziller). *E and F are mutually inverse homology isomorphisms.*

Def. *We define the cross product (sometimes called the external product) $\times : H^i(X) \otimes H^j(Y) \to H^{i+j}(X \times Y)$.*

Let $f \in C^i(X), g \in C^j(Y)$ and let $\sigma \in C_i(X), \tau \in C_j(X)$. Define $(f,g)(\sigma,\tau) := f(\sigma)g(\tau)$. This gives a map $(C^i(X) \otimes C^j(Y)) \otimes (C_i(X) \otimes C_j(Y)) \to \mathbb{Z}$. If we let the map (f,g) be 0 if σ, τ are not in the right places, then we can extend this to $(C^i(X) \otimes C^j(Y)) \otimes (C_*X(X) \otimes C_*(Y))_{i+j} \to \mathbb{Z}$.

By adjointness of $\otimes$ to hom *and using F, we get a map $C^i(X) \otimes C^j(Y) \to C^{i+j}(X \times Y)$. This map is $\times$. Can check that the differential $\delta(f \times g) = (\delta f) \times g + (-1)^i f \times (\delta g)$, so this map works on homology.*

We can examine some of the properties of the cross product

- Cross product induces a cross product on cohomology. It is natural. If $f : X \to X', g : Y \to Y'$ then

$$\begin{array}{ccc} H^i(X) \otimes H^j(Y) & \xrightarrow{\times} & H^{i+j}(X \times Y) \\ {\scriptstyle f^* \otimes g^*} \uparrow & & {\scriptstyle (f \times g)^*} \uparrow \\ H^i(X') \otimes H^j(Y') & \xrightarrow{\times} & H^{i+j}(X' \times Y') \end{array}$$

- The cross product is bilinear, associative, and graded symmetric ie $\sigma \times \tau (-1)^{|\sigma| \cdot |\tau|} \tau \times \sigma$.

We now internalize this. The cross product gives a map $H^i(X) \otimes H^j(X) \to H^{i+j}(X \times X)$. We can take a diagonal map $\Delta : X \to X \times X$ by taking $x \to (x,x)$. On cohomology this induces map $H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$.

Def. *The **cup product** on cohomology is*

$$\smile : H^i(X) \otimes H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

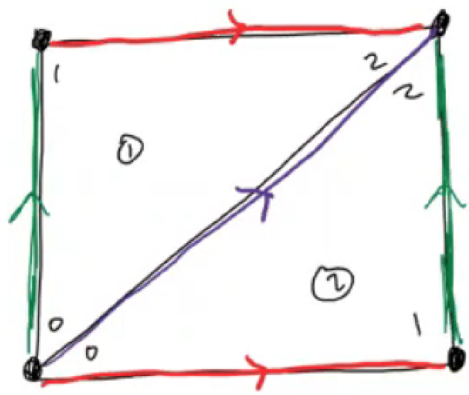
induces a map $H^*(X) \otimes H^*(X) \xrightarrow{\smile} H^*(X)$.

This is associative, graded symmetric and unital. Therefore, this is a ring structure on cohomology.

Def. *An explicit definition of the cup product in simplicial cohomology: $\alpha \in H^i(X), \beta \in H^j(Y)$. α is represented by $C_i(X) \to \mathbb{Z}$, β is represented by $C_j(X) \to \mathbb{Z}$. $\alpha \smile \beta$ should be defined by $C_{i+j} \to \mathbb{Z}$ $(\alpha \smile \beta)[v_0, \dots, v_{i+j}] := \alpha([v_0, \dots, v_i])\beta([v_i, \dots, v_{i+j}])$. More notes for CW/singular cohomology can be found in Hatcher 3.2.*

Remark. *It's not obvious that the two above definitions are the same, but they are.*

Ex. *Consider the torus*



Here 1 is supposed to be $\textcolor{cyan}{\blacksquare}$ and 2 is $\textcolor{orange}{\blacksquare}$ (Inna got it wrong in the initial lecture).

Note that two cycles are generated by $\textcolor{cyan}{\blacksquare} - \textcolor{orange}{\blacksquare}$. The cohomology class in degree 1 is $C^1 = \hom(C_1, \mathbb{Z})$ assigns to each map $[0,1] \to T^2$ an integer. This is a cocycle, when restricted to a boundary, it is 0. A cycle α has $\alpha(\textcolor{red}{\to}) + \alpha(\textcolor{green}{\to}) - \alpha(\textcolor{purple}{\to})$. Define α by $\alpha(\textcolor{red}{\to}) = \alpha(\textcolor{purple}{\to}) = 1, \alpha(\textcolor{green}{\to}) = 0$ and $\beta(\textcolor{red}{\to}) = 0, \beta(\textcolor{green}{\to}) = \beta(\textcolor{purple}{\to}) = 1$.

We see that $(\alpha \smile \beta)(\textcolor{cyan}{\blacksquare}) = \alpha(\textcolor{red}{\to})\beta(\textcolor{green}{\to}) = 1$ and $(\alpha \smile \beta)(\textcolor{orange}{\blacksquare}) = \alpha(\textcolor{green}{\to})\beta(\textcolor{red}{\to})$. $\alpha \smile \beta$ on a generator of H_2 is $(\textcolor{cyan}{\blacksquare} - \textcolor{orange}{\blacksquare}) = 1$.

By contrast, we see that $\beta \smile \alpha$ can be defined analogously, and $(\beta \smile \alpha)(\textcolor{cyan}{\blacksquare} - \textcolor{orange}{\blacksquare}) = -1$

Note that $H^*(T^2) = \mathbb{Z}[x, y](/x^2, y^2)$. People normally write $\Lambda(x) = \mathbb{Z}[x]/x^2$ where $|x|$ is odd (the degree of values has to be odd) and so we normally write $H^*(T^2) = \Lambda(x, y)/(xy + yx)$ (but people normally drop the $xy + yx$ in an abuse of notation).

Now, we are going to show that T is not homeomorphic with $S^1 \vee S^1 \vee S^2$.

Lemma. *$H^*(X \vee Y) \cong H^*(X) \oplus H^*(Y)$ for $n > 0$.*

This is due to

Theorem 3.8 (Weintraub 5.6.12, 5.6.13)**.**

1. *The cup product is bilinear, associative, graded symmetric. It is unit with $1^x \in H^0(X)$ and is the pullback along the map $X \to pt$ of the cochain $[pt] \to 1$.*
2. *The cup product is natural: for any $f : X \to Y$ then for any $\alpha, \beta \in H^*(Y)$, $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$. In particular, this is a ring homomorphism.*
3. *If $p : X \times Y \to X$ is the projection, then for any $\alpha \in H^*(X)$ then $p^*(\alpha) = \alpha \otimes 1^Y$.*

So we have

Proof of Lemma. Take a map $X \to X \vee Y \to X$ (the composition is the identity). This implies that we have $H^*(X) \to H^*(X \vee Y) \to H^*(X)$, where the composition is the identity. Also note that this holds for Y. By the universal coefficient theorem, and the fact that this is an isomorphism on homology, we get the desired. □

Note that this implies that nontrivial cup products in degree higher than 0 are the same as in the components. If $\alpha \in H^*(X), \beta \in H^*(Y)$ then $\alpha \smile \beta = 0$. This is because α is represented in $H^*(X \vee Y)$ as a cochain which is 0 on every singular simplex in Y (and the same for β).

This shows that $H^*(S^1 \vee S^1 \vee S^2)$ has generators in degrees $1, 1, 2$. All products are trivial, and so this is not the same as $H^*(T^2)$.

Ex. *$S^p \times S^q$ has homology $H_i(S^p \times S^q) \cong \begin{cases} \mathbb{Z} & \text{0, p, q, p + q} \\ 0 & \text{o.w.} \end{cases}$. This also holds for H^i by universal coefficient theorem.*

$H^0(S^p \times S^q)$ is generated by $1^{S^p \times S^q}$. $H^p(S^p \times S^q)$ is generated by $\alpha \otimes 1^{S^q}$ where α generates $H^p(S^p)$. Something similar holds for $H^q(S^p \times S^q)$ ie generated by $1^{S^p} \otimes \beta$.

Examining H^{p+q}. This is $\mathbb{Z}$. What is the relation with $(\alpha \otimes 1^{S^q}) \smile (1^{S^p} \otimes \beta)$. Note that this is equal to $\alpha \otimes \beta$ by computation. With field coefficients, we know that $H^*(S^p \times S^q) \cong H^*(S^p) \otimes H^q(S^q)$, but we don't have field coefficients.

Prop 3.1 (Weintraub 5.6.13)**.** *For $\alpha \in H^*(X), \beta \in H^*(Y)$. Then $\alpha \times \beta = p_1^*(\alpha) \smile p_2^*(\beta) = (\alpha \times 1^Y) \smile (1^X \times \beta)$ where p_1, p_2 are projections.*

Proof. Consider the diagonal $\Delta : (X \times Y) \to (X \times Y) \times (X \times Y)$. By definition, we see that $p_1^*(\alpha) \smile p_2^*(\beta) = \Delta^*(p_1^*(\alpha) \times p_2^*(\beta)) = \Delta^*((p_1 \times p_2)^*(\alpha \times \beta)) = ((p_1 \times p_2)\Delta)^*(\alpha \times \beta) = \alpha \times \beta$ since $(p_1 \times p_2)\Delta = 1$. $\square$

In our example, this implies that $\alpha \otimes \beta = \alpha \times \beta \in H^n(S^p \times S^q)$. This implies that $H^*(S^p \times S^q) = \Lambda(x_p) \otimes \Lambda(x_q)$ where $|x_p| = p, |x_q| = q$ modded out by $(x_p x_q) - (-1)^{pq} x_q x_p = \Lambda(x_p, x_q)$. This also works for $p = q$.

We get that

Prop 3.2. *For $p, q > 0$, $S^p \vee S^q \vee S^{p+q}$ is not homotopy equivalent to $S^p \times S^q$ by analyzing the cohomology ring.*

Ex. *$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$ for $|x| = 1$. $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ for $|x| = 2$.*

We first show $\mathbb{RP}^n$. Note that as a corollary, $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$.

We do this by induction.

- $n = 1$ has $\mathbb{RP}^1 = S^1$.
- Inductively, we have $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$ induces an isomorphism in cohomology for degree $\leq n-1$. We only need to worry about the degree n part.

  Define $r : \mathbb{RP}^{n-1} \times \mathbb{RP}^1 \to \mathbb{RP}^{2n-1}$. This maps $[x_0 : \cdots : x_{n-1}] \times [y_0 : y_1] \to [x_0 y_0 : x_1 y_0 : \cdots : x_{n-1} y_0 : x_0 y_1 : \cdots : x_{n-1} y_1]$. Let $\beta \times H^1(\mathbb{RP}^{2n-1}; \mathbb{Z}/2)$ be the generator. We know that $\mathbb{RP}^{n-1} \times \mathbb{RP}^1$ is a structure we know.

  Claim: $\beta^n \in H^n(\mathbb{RP}^{2n-1}; \mathbb{Z}/2)$ is a generator (ie nonzero). It suffices to find some map into $\mathbb{RP}^{2n-1}$ s.t. the pullback of β is nonzero. We will use r.

  Consider $p : \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1} \times \{[1:0]\} \hookrightarrow \mathbb{RP}^{n-1} \times \mathbb{RP}^1 \xrightarrow{r} \mathbb{RP}^{2n-1}$. $p^*(\beta)$ is a generator of $H^1(\mathbb{RP}^{n-1})$. By inductive hypothesis $(p^*(\beta))^{n-1}$

is the generator of $H^{n-1}(\mathbb{RP}^{n-1}) = p^*(\beta^{n-1})$. $q : \mathbb{RP}^1 \to \{[1 : 0 : \cdots : 0]\} \times \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^{n-1} \times \mathbb{RP}^1 \xrightarrow{r} \mathbb{RP}^{2n-1}$. $q^*(\beta)$ is a generator of $H^1(\mathbb{RP}^1)$.

We have $H^{n-1}(\mathbb{RP}^{n-1}) \otimes H^0(\mathbb{RP}^1) \leq H^{n-1}(\mathbb{RP}^{n-1} \times \mathbb{RP}^1)$ and the LHS is generated by $p_1^*(p^*(\beta)^{n-1})$ and we see that $H^0(\mathbb{RP}^{n-1}) \otimes H^1(\mathbb{RP}^1) \leq H^1(\mathbb{RP}^{n-1} \times \mathbb{RP}^1)$ generated by $1 \times p_2^*(q^*(\beta))$. By Kunneth, generator of $H^n(\mathbb{RP}^{n-1} \times \mathbb{RP}^1)$ is $p_1^*(p^*(\beta)^{n-1}) \times p_2^*(q^*(\beta)) = (p_1^*(p^*(\beta)^{n-1}) \times 1) \smile (1 \times p_2^*(q^*(\beta)))$. The first part is $r^*(\beta^{n-1})$ and the second part is $r^*(\beta)$, so this implies that the entire value is $r^*(\beta^n)$. Since this is a generator, this implies that β^n is nonzero/a generator (as desired).

Now $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{2n-1}$ induces an isomorphism in cohomology up to degree n. This implies that $H^*(\mathbb{RP}^n) \cong \mathbb{Z}/2[x]/x^{n+1}$.

Corrollary. *Let $n > m \geq 1$. If $f : \mathbb{RP}^n \to \mathbb{RP}^m$ is any map, then f_* is the 0 map. This is because we can analyze $f^* : \mathbb{Z}/2[x]/x^{m+1} \to \mathbb{Z}/2[x]/x^{n+1}$, and since $n > m$ this must be 0.*

Corrollary. *$H^*(\mathbb{RP}^\infty)$. Let $\alpha \in H^1(\mathbb{RP}^\infty)$ as the generator. Then we see that α^n must generate $H^n(\mathbb{RP}^\infty)$ (by inclusion of $\mathbb{RP}^n$ since the induced maps are isomorphism up to degree n).*

3.4 De Rham Cohomology

Consider a manifold and the differential forms on a manifold. Let F_k be the k-forms. Note that we have a cochain complex

$$F_0 \xrightarrow{d} F_1 \xrightarrow{d} F_2 \xrightarrow{d} \dots$$

since $d^2 = 0$. This is called the **de Rham cohomology**. There exists a map $F_k \to \hom(C_k(\mathcal{M}), \mathbb{R})$. This takes $\alpha \to (\text{chain} \to \int_{\text{chain}} \alpha)$. Fact: this induces an isomorphism on cohomology.

Exercise: what is Stoke's Theorem here?

Fact: a closed manifold (w/boundary) is a CW complex. This comes from Morse Theory.

Def. *A n-**manifold** X is a topological space s.t. for all points $x \in X$ there exists a nbhd $U \ni x$ s.t. $U \cong \mathbb{R}^n$. An **atlas** on X is a family of open sets $\{U_d\}$ which cover X and are charts. Construct a manifold from an atlas $\{U_\alpha\}_{\alpha \in A}$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \subseteq U_\alpha \to U_\alpha \cap U_\beta \subseteq U_\beta$.*

A ***manifold with boundary*** *is homeomorphic to* $R^{n-1} \times [0, \infty)$*.*

A ***closed manifold*** *is compact and has no boundary.*

Lemma.

1. *If* $(M, \partial M)$*, then* $M - \partial M$ *is a n-manifold and* ∂M *is a* $n-1$ *manifold.*
2. *If* M *is a n-manifold, then* $H_k(M) = H^k(M) = 0$ *for* $k > n$*.*
3. *If* M *is a compact n-manifold w/boundary, then* $H_k(M)$ *and* $H^k(M)$ *are finitely generated.*

Lemma. *If* M *is a n-manifold,* $x \in M$*, then* $H_n(M, M - x; G) \cong G$*. This can be done with excision, ie* $H_n(M, M - x; G) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; G) \cong G$*.*

Now, we introduce one of the most fundamental theorems in algebraic topology, Poincaré Duality. Due to time, we will skip a lot of technical details, but it is important to see this. The details can be found in Weintraub 5.6.7, 5.6.11, 6.

Def. *We construct the slant map. Consider* $C^j(X) \otimes C_j(X) \otimes C_k(Y) \to C_k(Y)$ *given by* $f \otimes \alpha \otimes \beta \to f(\alpha)\beta$ *(note that* $f(\alpha)$ *is an integer so this typechecks. The induces the* ***slant map*** $\backslash : H^j(X) \otimes H_{j+k}(X \times Y) \to H_k(Y)$ *as* $H_{j+k}(X \times Y) = (H_j(X) \otimes H_k(Y)) \oplus \left(\bigoplus_{i \neq j} H_i(X) \otimes H_{j+k-i}(Y)\right)$ *and the slant map is acts on* $H^j(X) \otimes H_j(X) \otimes H_k(Y)$ *as given above and* 0 *on the other terms.*

This is natural since if $f : X \to X', g : Y \to Y'$ *then* $g_*(f^*u \backslash v) = u \backslash (f \times g)_*(v)$*. It is also bilinear.*

Def. *Define* $\frown : H^j(X) \otimes H_{j+k}(X) \to H_k(X)$ *given by* $x \frown y := x \backslash \Delta_* y$*.*

In terms of simplices, $\varphi \in C^j(X), \sigma \in C_{j+k}(X)$*, then* $\varphi \frown \sigma = \varphi(\sigma|_{[v_0,\ldots,v_j]})(\sigma\ |_{[v_j,\ldots,v_{j+k}]}\ ) \in C_k(X)$*.*

Theorem 3.9 (Poincaré Duality)**.** *Let* M *be a closed* G*-oriented n-manifold and let* $[M] \in H_n(M; G)$ *be the fundamental class. Then the map*

$$\frown[M] : H^j(M; G) \to H_{n-j}(M; G)$$

is an isomorphism for all j*.*

Corrollary. *If G is a field, then $H^j(M;G) \cong H_j(M;G) \cong H^{n-j}(M;G)$ by universal coefficient theorem and PD.*

Corrollary. *If PD holds, then $H^j(M;G) = 0$ for all $j > n$. Same for homology.*

Ex.

- $\mathbb{RP}^n$ *and* $G = \mathbb{Z}/2$. $H^i(\mathbb{RP}^n) = \mathbb{Z}/2$ *for all* $i \in [n]$.
- T^2 *and* $G = \mathbb{Z}$. *This is also symmetric.*

Ex. *We are going to use PD to compute cup product structure on $\mathbb{RP}^n$. Suppose α is a generator of $H^1(\mathbb{RP}^{n+1})$.*

$$\frown[\mathbb{RP}^{n+1}] : H^1(\mathbb{RP}^{n+1}) \cong H_n(\mathbb{RP}^{n+1}) \cong H_n(\mathbb{RP}^n)$$

We see that, if ev *is the evaluation map $H^j(X;G) \otimes H_j(X;G) \to G$, then*

$$1 = \mathrm{ev}(\alpha^n, [\mathbb{RP}^n]) = \mathrm{ev}(\alpha^n, \alpha\frown[\mathbb{RP}^{n+1}]) = \mathrm{ev}(\alpha^{n+1}, [\mathbb{RP}^{n+1}])$$

We see that 2^{n+1} is a generator of $H^{n+1}(\mathbb{RP}^{n+1})$. Here, we are using the fact that cap product is adjoint to the cup product, which follows from definition.

Theorem 3.10. *Let M be a closed n-manifold with odd n. Then the Euler characteristic of M is 0.*

Proof. All manifolds are $\mathbb{Z}/2$ orientable. We see that

$$\chi(M) = \sum_i (-1)^i \mathrm{rank} H_i(M) = \sum_i (-1)^i \dim_k H^i(M;k)$$

With $\mathbb{Z}/2$ coefficients, we can apply PD and we see that

$$\dim H^i(M;\mathbb{Z}/2) = \dim H^{n-k}(M;\mathbb{Z}/2)$$

so $\chi(M) = 0$. □

Theorem 3.11 (Poincare Duality). *Let $(M, \partial M)$ be a compact oriented n-manifold w/boundary with fundamental class $[M] \in H_n(M)$. Then $\frown [M] : H^j(M, \partial M) \to H_{n-j}(M)$ is an isomorphism.*

Theorem 3.12. *Let M be a closed n-manifold with odd Euler characteristic: then M is not the boundary of an $n+1$-manifold.*

Proof. $\mathbb{Z}/2$ coefficients. Assume $M = \partial X$. We have a LES in homology

$$0 \to H_{n+1}(X) \to H_{n+1}(X, M) \to H_n(M) \to H_n(X) \to \cdots \to H_0(X, M) \to 0$$

The sequence is exact, so alternating sum of dimensions (as $\mathbb{Z}/2$ v.s.) is 0. Therefore, the sum of the dimensions is even. By PD, we see that $\sum \dim H_k(X) = \sum \dim H_{n+1-k}(X, M)$. Because $\chi(M)$ is odd, then $\sum \dim H_k(M)$ is odd. Therefore the sum of the dimensions in the LES is odd, which is a contradiction. $\square$

We can also construct the fundamental class. $[M]$ is a generator $H_n(M)$ and it is related to the orientation.

Ex. $H_2(T^2) = \mathbb{Z}$. *Choosing a generator ± 1 is choosing an orientation on the 2-cell.*

Ex. *Consider $\mathbb{RP}^2$. Note that $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}/2, H_2 = 0$ with $\mathbb{Z}$ coefficients. Since the top is 0, this implies that $\mathbb{RP}^2$ is not $\mathbb{Z}$-orientable. With $\mathbb{Z}/2$-coeff, we can choose a generator.*

In general, n-manifolds have one n-cell. If it is orientable, we can choose an orientation on it and it will define a cycle. If it is nonorientable: it will be a cycle.

On $\mathbb{RP}^2$, the top cell is attached with a 2 map, so it is not a cycle. Conversely, on T^2 the top cell is attached with a 0 map, so it is a cycle. On the Klein bottle, the 2-cell is attached orientably on one 1-cell, but non-orientably on the other.

Consequence: Suppose X is a manifold that has a oriented submanifold $N \subseteq X$ where $\dim N = i$. There is an inclusion $H_i(N) \xrightarrow{f_*} H_i(X)$, $[N]$ gives a homology class in $H_i(X)$. We say that N represents this class $f_*[N]$. If we assume that everything is orientable. If N represents a class in $H_{n-i}(X)$, then N also represents the corresponding class $H^i(X)$.

If $\alpha \in H^i(X)$ is represented by a codim i submanifold A and $\beta \in H^j(X)$ is represented by a codim j submanifold B, what represents $\alpha\smile\beta$. If A and B intersect transversely, $\alpha\smile\beta$ is represented by $A \cap B$.

We now go over orientations (Inna does not like this stuff). The idea is that a n-manifold M is G-orientable if $H_n(M; G) \cong G^{\#\text{connected components}}$. $[M]$: pick a generator for each connected component, add them up. Usually $G = \mathbb{Z}$ or $\mathbb{Z}/2$. If $G = \mathbb{Z}/2$ all n-manifolds are orientable. A good reference is Hatcher p242.

Proof of Poincaré Duality. We will assume G is a ring.

The hope is that a compact manifold is a union of finitely many convex open subsets of $\mathbb{R}^n$ s.t. the interiors are also convex open subsets of $\mathbb{R}^n$. Ideally, we would prove this for $\mathbb{R}^n$ and use Mayer-Vietoris and induction. Suppose there exists a modification of cohomology H_c^* for which the theorem worked (even if we ignored the closed condition). Also need a reverse to M-V for H_c^* so that the following diagram commutes

$$\begin{tikzcd}
H_c^i(U\cap V) \arrow[r] \arrow[d, "\frown[U\cap V]"] & H_i^c(U)\oplus H_c^i(V) \arrow[r] \arrow[d, "\frown[U]\oplus\frown[V]"] & H_c^i(U\cup V) \arrow[r] \arrow[d, "\frown[U\cup V]"] & H_c^{i+1}(U\cap V) \arrow[d, "\frown[U\cap V]"] \\
H_i(U\cap V) \arrow[r] & H_i(U)\oplus H_i(V) \arrow[r] & H_i(U\cup V) \arrow[r] & H_i(U\cap V)
\end{tikzcd}$$

This would give us an induction on which we can prove everything.

But, alas, this isn't true. We instead have to work around this. The outline is given by

1. Define H_c^*, where this is covariant on inclusions of opens and agree on H^* on compact spaces.
2. Prove PD for $\mathbb{R}^n$ (and also for convex open in $\mathbb{R}^n$).
3. Prove Mayer-Vietoris and the diagram commutes.
4. Use induction to prove for finite unions, use this prove arbitrary unions.

Orientability implies $[M]$ exists.

H_c^* is compactly supported cohomology. This is not cohomology (ie doesn't satisfy ES axioms).

Ex. *X simplicial complex. "Locally compact" means that every simplex is a face of only finitely many simplices. Ex: $\mathbb{R}^1$ with integer vertices and 1-simplices $[n, n+1]$. A non-examples is a point with infinitely many segments extending out from it.*

Define $C_c^i(X) :=$ maps{i-simplicies $\to G$} which are nonzero on only finitely many simplices. This is a well defined cochain complex, as if $\sigma \in C_c^i(X)$ is nonzero on only finitely many simplices, so is $\delta\sigma$. We can define $H_c^i(X) = H^i(C_c^*(X))$.

Ex. *$X = \mathbb{R}^1$. $C_c^0(\mathbb{R})$ is generated by maps which are nonzero on exactly one vertex. If $\delta\varphi = 0$, then φ is equal on both ends of any 1-simplex so $H_c^0(\mathbb{R}) = 0$ because the only such φ is uniformly zero. Now, consider $H_c^1(\mathbb{R})$. Note that 1-chains are cycles. $C_c^1(X) \to \mathbb{Z}$ (now we use $\mathbb{Z}$) by sending $\varphi \to \sum_{1 simplices} \varphi(\sigma)$. This is zero on any coboundary, so it's well defined on H^*. This induces $H_c^1(\mathbb{R}) \cong \mathbb{Z}$.*

We can define this on singular cohomology. Let $C_c^i(X) = \operatorname{colim}_{\text{compact } K \subseteq X} C^i(X, X - K)$. This is a nice colimit since it is filtered (since we can compare all objects as $K \cup L$ is compact). We see that $H_c^i(X) \cong \operatorname{colim}_{\text{compact } K \subseteq X} H^i(X, X - k)$.

Ex. *$X = \mathbb{R}^n$. K_i is a closed n-ball of radius i around origin. It suffices to consider these. $H_c^m(\mathbb{R}^n) \cong_i H_c^m(X, X - k_i)$. By excision $H^m(X, X - K_i) \cong H^m(D^n, \partial D^n) = \widetilde{H}^m(S^n)$. We therefore see that $H_c^m(\mathbb{R}^n) = \mathbb{Z}$ if $m = n$ and 0 otherwise.*

Now, we do M-V. Suppose U, V open and $X = U \cup V$. $K \subseteq U, L \subseteq V$ compact. Since X is orientable (ie $[X]$ exists) then there exist fundamental classes $\mu_K \in H_n(U, U - K)$ and $\mu_L \in H_n(V, V - K)$ by restriction. Because X is orientable, they agree inside $H_n(X, X - (K \cup L))$.

We have the following CD

$$\begin{tikzcd}
H^k(X, X - K \cap L) \arrow[r] \arrow[d, "\cong"] & H^k(X, X - k) \oplus H^k(X, X - L) \arrow[r] \arrow[d, "\cong"] & H^k(X, X - K \cup L) \arrow[dd, "\frown\mu_{K\cup L}"] \\
H^k(U \cap V, U \cap V - K \cap L) \arrow[d, "\frown\mu_{K\cap L}"] & H^k(U, U - K) \oplus H^k(V, V - L) \arrow[d, "\frown\mu_K \oplus \frown\mu_L"] & \\
H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U \cup V)
\end{tikzcd}$$

We can take the colimit over K, L to give the well-defined diagram (as shown in hw)

$$\begin{tikzcd}
H_c^k(U \cap V) \arrow[r] \arrow[d] & H_c^k(U) \oplus H_c^k(V) \arrow[r] \arrow[d] & H_c^k(X) \arrow[d] \\
H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(X)
\end{tikzcd}$$

So we have shown that if M is a union of finitely many convex open subsets in $\mathbb{R}^n$ then it has PD. The final step (extension to arbitrary unions) is in Hatcher, for easier than the others. This is due to the fact that cohomology commutes with filtered colimits (even the compact case). $\square$

4 Homotopy Theory

I stopped transcribing around this time.